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Application of the Hurst coefficient, calculated with the use of the Siroky method on financial markets

Abstract

The paper analyses the Hurst exponents calculated with the use of the Siroky method in two time intervals of 625 and 1250 sessions for the group of 570 financial instruments (Warsaw Stock Exchange equities – 320, equity indexes – 7, commodities – 41, and FX market – 135). The study also covers an analysis of the normality of the distribution of logarithmic rates of return, and the verification of statistical hypotheses with the use of the following statistical tests: Jarque-Bera (JB), Shapiro-Wilk (SW), and d'Agostino-Pearson (DA).

In the second part of the paper, the change of the Hurst coefficient over time was analysed, while in the third part two linear regressions of the form $H(t) = a + m \cdot t$ were performed for each of the analysed assets, as well as the determination factor R^2. This part of the study aims to answer the question whether the slope of the regression line has a positive or negative value and what the quality of such a fit is with the use of linear regression. Such an analysis enables to observe changes in the fractal dimension, and thus the risk in financial markets over a long period of time.

The main conclusion that was drawn from the research may be formulated as follows: the value of the *H* exponents decreased in the analysed time windows, which means an increase in the fractal

dimension (d), and thus the investment risk in financial markets. The obtained results can be used in the process of constructing an investment portfolio in financial markets.
The research is part of the ongoing discussion on the effectiveness of financial markets.

Keywords: Hurst exponent, market efficiency, developed countries
JEL Classification Codes: G10, G14

Introduction

There is a wide spectrum of mathematical tools constructed to analyse processes characterised by long-term data dependence. One of the important turning points in long-term data series analysis was a method of scaled range analysis, proposed by Hurst [1951, pp. 770–799]. The starting point in Hurst's work was Einstein's study of Brownian motions [Einstein, 1908, pp. 469–502], in which the relationship of the distance r covered by the molecule during time t was expressed in the following form:

$$r = c \cdot \sqrt{t}$$

where c represents a constant.

This equation mainly concerns the case when the sequence of increments of the molecule's path in time is a random walk, characterised by the independent random variables normally distributed [Weron A., Weron R., 2015, pp. 59–75; Kapecka, 2015, pp. 59–75]. Meanwhile, the results obtained by Hurst led to the conclusion that the overwhelming number of natural phenomena (temperature changes, river and sea floods, atmospheric precipitation, sunspot activity) are not subject to a random walk, but constitute the so-called processes with long-term memory, which were later called fractional Brownian motion, representing the superposition of trend and noise [Peters, 2003, pp. 37–38; Mastalerz-Kodzis, 2003, pp. 37–38].

For the scaled range method, the Hurst exponent is defined by the following formula [Peters, 1997, p. 64]:

$$\left(\frac{R}{S}\right)_n = a \cdot n^H \tag{1}$$

R – range of fluctuations,
S – standard deviation,
$\left(\frac{R}{S}\right)_n$ – so-called scaled range,
$a > 0$ – positive constant,
$n \in N$ – number of time series observations.

Calculating the logarithm of both sides of the equation (1) leads to the following expression for $ln\left(\frac{R}{S}\right)_n$:

$$ln\left(\frac{R}{S}\right)_n = H \cdot ln(n) + ln(a) \quad (2)$$

So, in order to calculate the Hurst exponent, first it is necessary to estimate the scaled range $\left(\frac{R}{S}\right)_n$ for different n, and then solve equation (2) using linear regression. The Hurst exponent is a regression coefficient, estimated with the use of the least squares method. This method was further improved by Mandelbrot and Wallis [1968].

To determine the value $\left(\frac{R}{S}\right)_n$ there can also be used the method of dividing the time series $X_1, X_2, \dots X_n$, into k time sub-series of n elements each, so that $N = k \cdot n$, where n, k are natural numbers. In addition, a condition regarding n and N is applied: $10 \le n \le \frac{N}{2}$. Next, on the basis of the arithmetic mean and standard deviation of each sub-series, the cumulative deviation and the range of cumulative time series are determined in order to obtain the so-called scaled stretch marks. The procedure is repeated for different lengths of the time series n. Consequently, the dependence of the size $\left(\frac{R}{S}\right)_n$ and the length of the series n is obtained. This procedure is described in detail in Kowgier's paper [2009, pp. 157–167]. A lot of work has been dedicated to the analysis of scaled coverage in various fields [De La Fuente et al., 1998, pp. 95–100; Alvarez-Raminez et al., 2008, pp. 6452–6462] One of the basic disadvantages of the $\left(\frac{R}{S}\right)_n$ method is its sensitivity to the number of observations n. This issue was addressed in the works of: Couillard and Davison [2005, pp. 404–418], Anis and Lloyd [1976, pp. 111–116], Taqqu and Teverovsky [1998. pp. 177–217].

The first formula modifying the Hurst pattern in terms of its use for small numbers n was proposed by Anis and Lloyd [1976, pp. 111–116]:

$$\left(\frac{R}{S}\right)_n = \begin{cases} \dfrac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot \sum_{k=1}^{k=n-1} \sqrt{\dfrac{n-k}{k}} & \text{dla } n \le 340 \\ \sqrt{n \cdot \dfrac{\pi}{2}} \cdot \sum_{k=1}^{k=n-1} \sqrt{\dfrac{n-k}{k}} & \text{dla } n > 340 \end{cases} \quad (3)$$

Formula (3) proved to approximate $\left(\frac{R}{S}\right)_n$ values better than the Hurst formula, but for *n* higher than 340 there were still significant discrepancies between $\left(\frac{R}{S}\right)_n$ calculated with the use of Anis and Lloyd's formula and the mean values $\left(\frac{R}{S}\right)_n$ computed with the computer simulations [Kapecka, 2015, pp. 59–75]. In 1994, Peters corrected formula (3), which significantly improved the calculation results for any *n* [Peters, 1994]:

$$\left(\frac{R}{S}\right)_n = \begin{cases} \dfrac{n-\frac{1}{2}}{n} \cdot \dfrac{\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot \sum_{k=1}^{k=n-1} \sqrt{\dfrac{n-k}{k}} & \text{dla } n \leq 340 \\ \dfrac{n-\frac{1}{2}}{n} \cdot \sqrt{n \cdot \dfrac{\pi}{2}} \cdot \sum_{k=1}^{k=n-1} \sqrt{\dfrac{n-k}{k}} & \text{dla } n > 340 \end{cases} \tag{4}$$

Depending on the needs, methods determining the Hurst exponent can be used to study one-dimensional time series:

a) Long range analysis method (R/S) (described above) [Hurst, 1951, pp. 770–799],
b) *Detrended Moving Average* (DMA) [Alessio et al., 1995, pp. 197–200; Mantenga, Stanley, 1995, pp. 46–49],
c) *Detrended Fluctuation Analysis* (DFA) [Bunde, Havlin, 1995, pp. 97–100; Peng et al., 1994],
d) *Generalized Hurst Exponent* (GHE) [Di Matteo, 2003, pp. 183–188],

or multidimensional:

- *Multifractal Detrended Fluctuation Analysis* (MF-DFA), [Abry, Veitch, 1998; Kantelhardt et al., 2002],
- *Wavelet Transform Module Maxima (*WTMM) [Muzy et al., 1994, pp. 245–302].

Another method for determining the Hurst coefficient is the box method, which consists in covering the graph of the analysed function with a square grid with a certain side and counting the squares having at least one point in common with the examined graph. Then the grid is considered with a side that is *k* times smaller and the squares having at least one point in common with the analysed graph are counted again (Voss, 1991, pp. 816–817; Mastalerz-Kodzis, 2003]. This method was applied, among others, by Ehlers to create the Fractal Adaptive Moving Average [Ehlers, 2005, pp. 81–82]. In turn, Ehlers and Way modified the box method to include not only closing prices, but also the highest and lowest prices in the analysed period [Ehlers, Way, 2010, pp. 16–20].

There is a close relationship between the Hurst exponent and Minkowski's fractal dimension (*d*) [Kowgier, 2009, pp. 157–167]:

$$d = 2 - H \tag{5}$$

The proof of this relationship with the use of the fractional Brownian motion and self-similarity of the stochastic process was presented in the work of Kowgier [2009, pp.157–167]. Estimation of the fractal dimension is justified because it enables to gain additional knowledge about the behaviour of prices and rates of return, and thus contributes to a better understanding of their nature.

The method of calculating the Hurst exponent proposed by Siroky [2017, pp. 18–21 and 45] directly refers to the method of calculating the fractal dimension using the slope coefficient of sub-segments of the analysed time window, whose authors are Ehlers and Way [2010, pp. 16–20]. According to the method of segment division, the analysed time series should be divided into two segments of equal data length, with S_2 as the slope factor in the current data segment and by S_1 in the previous one. The slopes S_2 and S_1 are defined as follows [Siroky, 2017, pp. 18–21 and 45]:

$$S_2 = \frac{H_2 - L_2}{1} = H_2 - L_2$$

$$S_1 = \frac{H_1 - L_1}{1} = H_1 - L_1$$

where:
H_1 and H_2 – the highest price in the first and second data segments, respectively, L_1 and L_2 – the lowest price in the first and second data segments, respectively. For the entire analysed range, the slope factor will be equal to [Siroky, 2017, pp. 18–21 and 45]:

$$S_{tot} = \frac{H_{tot} - L_{tot}}{2}$$

where:
H_{tot} and L_{tot} – the highest and lowest prices in the entire data segment, respectively.

In turn, the fractal dimension calculated for all data included in the analysed range will be equal to [Siroky, 2017, pp. 18–21 and 45]:

$$d = \frac{ln(S_1 + S_2)}{ln(2)} - \frac{ln(S_{tot})}{ln(2)}$$

By dividing the observed range of data into smaller and smaller periods, reaching data for a single session, the slope coefficient becomes equal to the *TR* (*True Range*) value proposed by Wilder and equal to the difference between the highest and lowest price in a given session [Wilder, 1978, pp. 22–35]. With this approach, the value of the slope factor for all the observed sessions is equal to the value of *R*, divided by the number of sessions (*n*), where *R* is equal to the difference between the highest and lowest price in the analysed time window. Then the expression for the fractal dimension *d* will take the form [Siroky, 2017, pp. 18–21 and 45]:

$$d=\frac{ln\sum TR-ln\frac{R}{n}}{\ln n}=\frac{ln\left(\frac{\sum TR}{\frac{R}{n}}\right)}{\ln n}$$

Noting that $\sum TR$ is equal to the product of the Mean True Range value (MTR) and the number of observations (n), the expression for the fractal dimension is as follows [Siroky, 2017, pp. 18–21 and 45]:

$$d=\frac{\ln\left(\frac{MTR\cdot n}{\frac{R}{n}}\right)}{\ln n}=\frac{\ln\left(\frac{MTR\cdot n^2}{R}\right)}{\ln n}=\frac{2\cdot\ln n-\ln\left(\frac{R}{MTR}\right)}{\ln n}=2-\frac{\ln\left(\frac{R}{MTR}\right)}{\ln n}$$

Based on the relationship (5), the value of the Hurst coefficient is determined by the equation [Siroky, 2017, pp. 18–21 and 45]:

$$H=\frac{\ln\left(\frac{R}{MTR}\right)}{\ln n}$$

The MTR value can also be obtained as the Average True Range indicator, averaged over a given time window, with the latter being calculated on the basis of daily data. Siroky proposes the following way of applying the ATR indicator [2017, pp. 18–21 and 45]:

$$MTR\approx\frac{1}{n}\cdot\sum_{t=1}^{t=n}ATR_t$$

$$ATR_t=Max\left(H_t-L_t,\,|H_t-C_{t-1}|,\,|L_t-C_{t-1}|\right)$$

where:

H_t – highest price on the session t,

L_t – lowest price on the session t,

C_{t-1} – closing price on the session t–1.

The clear advantage of Siroky's method is applying not only closing prices, but also the highest and lowest prices in the calculation process, which should be considered as a method more adequate for use in financial markets than others. Unlike the scaled range method, the Siroky method does not impose the condition that the number of session n has many divisors. Therefore, this method should be considered more appropriate for use in financial markets than others. The Siroky method can, therefore, be used for shorter time intervals, while other methods often require longer time horizons. One drawback of the Siroky method is the use of the approximate MTR value. This approximation becomes negligible for large n. Regardless of the type of time series studied, the Hurst exponent assumes values in the range (0; 1), while obtaining boundary values is extremely difficult, therefore, they are often considered theoretical. Depending on the value of the Hurst exponent, the following interpretations of the time series are given [Barunik, Kristoufek, 2010, pp. 3844–3855]:

a) $H \in (0; 0.5)$ – the time series is defined as anti-persistent, characterised by high variability and high probability of frequent changes in the direction of short-term trends.
b) $H = 0.5$ – the time series does not have a dominant trend, which means that subsequent changes take on a random nature (random walk). The probability of both changing the trend and maintaining the current trend is the same and equal to 0.5.
c) $H \in (0.5; 1)$ – the time series is defined as persistent, characterised by low variability and low probability of frequent changes in the direction of short-term trends. In other words, it is a time series with an ordered course, which maintains the current trend (momentum). The higher the Hurst exponent, the higher the course orderliness.

It should be remembered that the Hurst coefficient only measures the general degree of occurrence of trends or their absence on financial markets, but it definitely fails to measure how prices on financial markets adjust after shocks.

The purpose of the study is to calculate the Hurst exponent with the use of the Siroky method for specific groups of assets: Equities listed on the WSE, Equity Indices, Commodities, Currency Market (pairs: USDXXX,[1] EURXXX, JPYXXX, Other Currency Pairs), as well as determining how these coefficients change over time. This is the first study proceeded on such a large scale with the application of Siroky's method.

Another but less important aim of the article is to verify the hypothesis about the normality of the distribution of rates of return of the analysed financial instruments as well as the normality of the distribution of average values of the Hurst coefficients calculated with the use of the Siorky method for the analysed instruments. The authors of the study are not aware of any scientific papers dedicated to the Hurst exponents method proposed by Siroky, nor calculating the variability of the Hurst coefficients obtained with the use of this method.

Literature review

Di Matteo et al. performed calculations of the Hurst exponent on the basis of the GHE method for the following market indices (the period covered by the analysis is given in brackets): NASDAQ 100 (1900–2001), NIKKEI 225 (1990–2001), WIG (1990–2001) and JSX (1991–2001), as well as 37 equity indexes (1997–2001). They proved that in the case of developed markets, the Hurst exponent values were lower than 0.5, while for less developed markets were higher than 0.5. In addition, by dividing the analysed period into groups of 3000 sessions, the authors indicated that the Hurst exponent values tended to fluctuate below 0.5;

[1] Contractual notation denoting the exchange rate of pairs in which USD is one of the currencies. An analogous convention was used for EURXXX and JPYXXX.

meanwhile, in the case of JSX and WIG indexes such fluctuations occurred mainly above 0.5 [Di Matteo et al., 2003, pp. 183–188].

Lipka and Los, analysing the daily data of eight equity indexes: ATX (Austria), KFX (Denmark), CAC 40 (France), DAX (Germany), OSLO TOTAL INDEX (Norway), IBEX 35 (Spain), MADRID GEN Index (SMSI – Spain) and FTSE 100 (the United Kingdom) for various time intervals (with the largest number of observations, i.e., 4437 collected for the British index, data for the period: April 2, 1984 to October 23, 2000), proved that the values of the Hurst exponent, calculated using the modified MF-DFA method, were lower than 0.5 for all the indices, except for the Danish stock exchange index, for which the Hurst exponent was equal to 0.55. For stock indices of Spain, Germany and France, the values of the Hurst exponent were slightly lower than 0.5: IBEX (0.46), SMSI (0.48), DAX (0.47) and CAC40 (0.46). On the other hand, the calculation of the Hurst exponent on the basis of the standard MF-DFA procedure[2] clearly confirmed that the values of all the Hurst exponents were lower than 0.5; wherein for some equity indexes these values were only slightly lower than 0.5: Austria (0.48), Norway (0.49), Spain IBEX 35 and SMSI (0.46 each) [Lipka, Los, 2003].

Kyaw et al., calculating the Hurst exponents for six Latin markets (Argentina, Brazil, Chile, Colombia, Mexico, and Venezuela) with the uses of the MF-DFA method, proved that Latin American equity markets are persistent, except for the Colombian and Mexican stock exchanges, which remain anti-persistent [Kyaw et al., 2004].

Barunik and Kristoufek, analysing one-minute, daily and monthly rates of return of the S&P 500 index for the period 1983–2009, using the following methods: R/S, GHE, MF-DFA, DMA, drew the conclusion that the best method for calculating the Hurst exponent is the GHE, regardless of the occurrence of so-called fat tails of the rates of return distributions. Their results clearly showed a change in the US stock market, which in the 1980s was characterised by strong persistence, while in the period 2005–2009 it became clearly anti-persistent [Barunik, Kristoufek, 2010, pp. 3844–3855].

Jud proved that in the period 1981 – July 2017, the value of the Hurst exponent, calculated for the six-month time intervals of the S&P 500 index, fluctuated around the value of 0.5. Thus, the author demonstrated the existence of two types of periods on the American market: the first in which short-term upward trends dominate and the second type of periods in which there are no such trends. In addition, the sharp decline in the Hurst exponent in 2015, the highest in twenty years, was recognised as one of the reasons for investors' rapid withdrawal of funds from hedge funds. It resulted in the fact that mathematical models for investing in line with the trend began to fail. Obviously, customers withdrawing funds had no idea about the historical and current values of the Hurst exponent, and they only pointed to the decrease in returns generated by hedge funds. In addition, the author concluded that the values of the Hurst exponent vary depending on the sector to which the given security or

[2] IDL Wavelet Toolkit calculation package, developed by Research Systems and available on the website: http://ion.researchsystems.com/IONScript/wavelet/website

stock index is included. In addition, the occurrence of cycles on the graph smoothed using a half-year double exponential moving average (DEMA) of the Hurst exponent was proved [Jud, 2017, pp. 71–75].

Kroha and Skoula constructed a technical analysis indicator based on the Hurst exponent (Moving Hurst) referring to another indicator (Polarised Fractal Efficiency). Further they indicated the hypothesis that the transaction system based on the Moving Hurst and Polarised Fractal Efficiency generated higher rates of return compared to the transaction system using the MACD (Moving Average Convergence Divergence) oscillator for NASDAQ and DAX indexes [Kroha, Skoula, 2019, pp. 371–378]. The authors applied low values of n = 16 and n = 32 to calculate the Hurst exponent [Hannula, 1994, pp. 38–41].

Raimundo and Okamoto, based on the Hurst exponent and scaled range (R/S) for such currency pairs as: AUDJPY, CHFJPY, EUROJPY, GBPJPY and EUROCHF for one-session, one-hour and fifteen-minute time horizons from the period 01.01.2003–30.12.2014, proved the possibility of use of the Hurst exponents as a tool for determining the degree of correlation and persistence in currency markets [Raimundo, Okamoto, 2018, pp. 116–124]. Their research confirmed earlier results obtained by Corazzy and Malliaris, who revealed that the values of the Hurst exponents statistically differ from 0.5 for most of the analysed currency pairs and change dynamically over time, concluding that changes in exchange rates of currency pairs are subject to the Brownian motion [Corazzy, Malliaris, 2008, pp. 387–401].

Kale and Butar certified that the distribution of 500 Hurst exponent values obtained on the basis of the R/ S method for the analysed time series resulted to be normal. In the process of analysis, the following tests were implemented: Shapiro-Wilk, Kolmogorov-Smirnov and Anderson-Darling. In all the cases, the obtained values of the p-value parameter were significantly greater than 0.05 [Kale, Butar, 2011, pp. 8–19].

The literature on the subject also indicates works devoted to the variability of the Hurst exponent. Ryvkina defined a class of Gaussian processes, extending the fractional Brownian motion for H > 1/2 by permitting arbitrary measurable functions H as variable Hurst parameters [Ryvkina, 2015, pp. 866–891]. Garcin improved the existing literature on estimating the time-dependent Hurst exponents by proposing a smooth estimate obtained by variational calculus. The method resulted to be quite promising on the FX market, in particular for the Hurst exponents higher than 0.5 [Garcin, 2017, pp. 462–479].

Research on the Hurst exponent was also conducted by Polish scientists. Wołoszyn confirmed that in the 1995–1998 period, the value of the Hurst exponent, calculated for Stomil and Elektrim stock prices were equal to 0.625 and 0.524, respectively. Computing the Hurst coefficient for different lengths of the time horizon (*n*), the author obtained the following results. For *n* belonging to the ranges from 5 to 20 sessions, from 21 to 50 sessions and from 51 to 100 sessions, the Hurst exponents were equal to: 0.630, 0.635 and 0.415, respectively [Wołoszyn, 2001, pp. 5–23].

Marcinkiewicz applied the Hurst exponent to analyse trends in the capital market and to implement this indicator in specific investment strategies on the basis of technical analysis in the period 1995–2006, using such tools as: moving averages, Momentum, Bollinger Bands, ADX, Channel Breakout, Commodity Channel Index. The following equity indexes were included in the analysis: WIG20, CAC40, DAX, DJIA, S&P 500, NIKKEI 225, NASDAQ as well as the EUROUSD. Calculations were conducted for = n 2800 on the basis of daily logarithmic rates of return. The author proved the relationship between the value of the Hurst exponent and the results obtained for investment strategies based on technical analysis. The rates of return received by transaction systems for time series with *H* close to 0.5, were lower than for transaction systems for time series with *H* values higher than 0.5 (except for the DAX index) [Marcinkiewicz, 2006, pp. 231–239].

Kowagier, on the basis of daily and weekly data for BRE share prices and WIG and WIG20 indices in the year 2005 and the length of daily data periods (two for each instrument) 74 and 50, 100 and 40, 110 and 40, respectively, and weekly 40 and 30, 42 and 12, 40 and 12, respectively, proved that the Hurst exponent values, calculated with the use of the R/S method were in all the cases higher than 0.5 (the lowest value was equal to 0.58 and was reported for daily data of BRE shares, and the highest was equal to 0.67 and was registered in two cases: for weekly data for BRE shares and for weekly data for the WIG index) [Kowagier, 2009, pp. 157–167].

Kapecka conducted an analysis of the results for the Hurst exponents calculated in the period from January 1993 to January 2013, broken down into developed and emerging markets. Developed markets include the following stock indexes (the Hurst exponent is given in brackets): DJIA (0.55), S&P 500 (0.58), DAX (0.56), Nikkei 225 (0.67), and Hang Seng (0.56). In turn, the following indices were classified into emerging markets: WIG20 (0.69), Bovespa (0.86), XU100 (0.64), Sensex 30 (0.57), and Shanghai Composite Index (0.56). The results of the research confirmed the occurrence of trends in both developed and emerging markets (H> 0.5). Based on the results of the research, the author contested that the economic situation in countries whose stock indexes were the subject of the analysis had the greatest impact on the value of the Hurst exponent [Kapecka, 2015, pp. 59–75].

Czarnecka and Wilimowska examined the WIG20 index for two time intervals (02.01.2014––17.10.2014 and 02.01.2015–16.10.2015), characterised by the Hurst exponent belonging to two different ranges. In each of the analysed time series, for all the analysed instruments, 200 subsequent changes in the price direction were noted (increase, decrease, increase, decrease, etc.). In selected time intervals, slight downward trends were observed, with the first of them being the anti-persistent series (H = 0.41), and the second one – persistent (H = 0.6) [Czarnecka, Wilimowska, 2018, pp. 45–72].

Rzeszótko, examining monthly logarithmic rates of return of shares of nine banks listed on the Warsaw Stock Exchange in the following period: date of their first listing – 29.04.2016 (end date), proved that in all the cases the values of the Hurst exponents were higher than 0.5. The highest value of the Hurst exponent was calculated for Millennium bank's rates of return and equal to 0.66; and the lowest for BPH shares' rates of return, when it amounted to 0.54 [Rzeszótko, 2016, pp. 131–141].

Buła surveyed time series of weekly logarithmic rates of return for future contracts from September 1998 to February 2015, for which the prices of 21 agricultural commodities were underlying instruments. Cycles of approximately one year were reviled for eight agricultural goods: corn, cotton, frozen orange juice, live cattle. pork, soybean, soybean meal and timber. In the period of one year, the price behaviour of these goods seemed to be random, however, for longer periods it was characterised by a significant level of anti-persistence. In turn, in the group of metals, future contract prices were characterised by persistence and the presence of 4-year cycles [Buła, 2015].

Methodology

The research was conducted in the following groups of assets (the number of assets from a given group after the dash):

a) Equities listed on the Warsaw Stock Exchange – 320;
b) Equity indexes – 74;
c) Commodities – 41;
d) Currency pairs divided into segments:
    1. USD exchange rate in relation to 42 other currencies (USDXXX);
    2. EURO exchange rate in relation to 41 other currencies (EURXXX);
    3. JPY exchange rate in relation to 40 other currencies (JPYXXX);
    4. Other currency pairs – 12 in total.

In total, prices of 570 financial instrument were analysed, see the Annex. The Annex contains information on the date of the first listing of individual instruments available in the databases.

In the group of Equities listed on the WSE, the selection criterion was the length of their listing on the secondary market – they should have been quoted for at least two and a half years before December 31, 2019. For other assets, the selection criterion was the availability of daily data in the following portals: stooq.pl, investing.pl and bossa.pl.

For each of the assets, the calculations were divided into the following stages:

1. Examination of the normality of the distribution of logarithmic rates of return over the periods:
    a) Date of the first listing of the instrument – 31.12.2019[3] (full time window);
    b) 625 sessions preceding the session on 31.12.2019[4] (625-sessions window);
    c) 1250 sessions preceding the session on 31.12.2019 (1250-session window)

    In this case, two hypotheses were formulated:

[3] For some assets, the last listing in 2019 took place on 30.12.2019, however, in the further part of this article, the last session of the year will be consistently designated as 31.12.2019.

[4] In the study of Jud, a 125-session interval was used, corresponding to the six-month horizon. Thus, in the case of the two and a half years and in case of five years' investment horizon, the following numbers of sessions should be applied: 625 and 1250, respectively [Jud, 2017, p. 74].

H_0: The distribution of return rates in the analysed time window is a normal distribution.
H_1: The distribution of return rates in the analysed time window is not a normal distribution.
Three tests were selected for verification of these statistical hypotheses: Jarque-Bera (JB), Shapiro-Wilk (SW) and d'Agostino-Pearson (DA).
The choice of logarithmic rates of return was dictated by their basic advantage, i.e., additivity.

2. Change of the Hurst exponent over time – calculation of the Hurst exponent values for 625- and 1250-session windows. In the following, the H exponents values for 625- and 1250-session windows will be designated: H_{625} and H_{1250}, respectively. Calculations of the H exponents were achieved according to the methodology proposed by Siroky. For the first 625 and 1250 prices of a given instrument, the H_{625} and H_{1250} were calculated for the first time (D_0),[5] after which the calculation window was moved by one session. The next values of the H_{625} and H_{1250} were given for sessions numbers: 2–626 and 2–1251, respectively. Then the calculation window was moved to the next session. In this way, for the financial instrument listed totally at K sessions, $K-650+1$ exponents of H_{625} and $H-1250+1$ exponents of H_{1250} were computed. In some cases, due to the shorter trading period of a given instrument, it was not possible to calculate the value of H_{1250} (the analysis did not include instruments, mainly from the equities listed on the WSE group for which H_{625} could not be achieved). For each of the instruments, the highest and lowest H values were given in the entire analysed period (for both calculation windows, i.e., 1250 and 625 sessions), the last value H, i.e. as of 31.12.2019 (for both calculation windows), as well as the value of H_{1250} five years earlier (i.e., at the end of the session falling 1250 sessions before 31.12.2019) and the value of H_{625} two and a half years earlier (i.e., at the end of the session falling 625 sessions before 31.12.2019). The average values of H_{1250} and H_{625} were also assessed for each analysed instrument.[6]
3. For each of the analysed assets, two linear regressions of the following form were implemented: $H(t)=a+m\cdot t$:
   a) The first for all the obtained H_{1250} and H_{625} (i.e., full time window);
   b) The second for H_{1250} and H_{625} calculated only for the last 1250 sessions (i.e., within approx. 5 years) preceding the end date of the study, i.e., 31.12.2019.

   The determination coefficients R^2 were also estimated.
   The purpose of this study is to answer the question whether the slope of the regression line has a positive or negative value and what is the quality of such a fit with the use of linear regression.
4. Because of the simple relationship between the fractal dimension (d) and the Hurst exponent (equation 5), the values of the former were calculated. In particular, the average values

[5] This date will be denoted as D_0. Obviously, for H_{1250} and H_{625} two different D_0 dates are obtained.
[6] Cajueiro and Tabak implemented in their model the so-called "rolling sample". That is, estimates were applied over a given data sample, the statistics were calculated, then the sample was moved up, or "rolled" forward one observation before the procedure was repeated. See Cajueiro, Tabak [2004] pp. 349–352.

of the fractal dimension were assessed for the 1250- and 625-session horizon, based on 1250 sessions preceding the end date of the analysis, i.e., 31.12.2019.

5. In the next part of the study, an analysis of the normality of the distribution of the average Hurst exponents (H_{1250} and H_{625}) was developed. The average values were calculated for each of the analysed financial instruments and then grouped: Equities listed on the WSE, Equity indices, Commodities, Currency market (broken down into 4 currency pair segments: USDXXX, EURXXX, JPYXXX and Other currency pairs). For this purpose, three types of tests were applied: Jarque-Bera (JB), Shapiro-Wilk (SW) and d'Agostino-Pearson (DA). Two statistical hypotheses were verified:
   $\boldsymbol{H_0}$: In a given segment of the financial market and in the analysed time window the distribution of average values of the Hurst exponents is a normal distribution.
   $\boldsymbol{H_1}$: In a given segment of the financial market and in the analysed time window the distribution of average values of the Hurst exponents is not a normal distribution.

Results

4.1. Examination of the normality of the distribution of logarithmic rates of return

As a result of the statistical tests, the following results were obtained:

a) In the case of the analysis of the normality of the rate of return distribution and the full time window, the null hypothesis was rejected in favour of the alternative hypothesis for all the analysed financial instruments.
b) In the case of the 625- and 1250-session window, the results are presented in Table 1. For the 625-sessions window, there were no reasons for rejecting the null hypothesis in the case of three companies listed on the Warsaw Stock Exchange (KGHM, MBANK and PKO BP), two Equity indices (PSEI and WIG), two commodities (cocoa and coffee (Arabica) and sixteen currency pairs (including six from the USDXXX segment, eight from the EURXXX segment, and two from the Other currency pairs segment). Only for the following currency pairs: USDDKK, USDEUR, EUREGP, EURIDR, EURNAD, and EURZAR, the p-value was higher than 0.05 for only one statistical test. In other cases, the p-value was greater than 0.05 for at least two different statistical tests. In turn, for the 1250-sessions window, there was no reason to reject the null hypothesis in the case of only two exchange rates: USDKRW and USDSDG.

The basic conclusion that can be drawn from the analysis of the distribution of rates of return is that the distribution of rates of return in financial markets is not a normal distribution. Analysing the percentage of cases in which there were no reasons to reject the null hypothesis for each of the analysed financial market segments, it can be concluded that the highest percentage when the rates of return were normally distributed were recorded in the Foreign exchange market segment and the lowest in the segment of Equities listed on the WSE. It can also be noted that in the case of a 625-session window, the number of cases when there was no reason to reject the null hypothesis was definitely higher than for a 1250-session window.

Table 1. Financial instruments for which there were no reasons to reject the null hypothesis for 625 and 1250 session windows (p-value higher than 0.05 in italics)

	Financial instrument name	Time horizon	*p-value* for three types of test			How many times p-value > 0.05
Equities listed on the WSE			JB	SW	DA	
1	KGHM	625	*0.2058*	*0.1359*	*0.2082*	3
2	MBANK	625	*0.5901*	*0.8395*	*0.5408*	3
3	PKO BP	625	*0.0545*	*0.1456*	*0.0878*	3
Equity indexes						
1	PSEI	625	*0.2265*	*0.1902*	*0.2429*	3
2	WIG	625	*0.0628*	*0.1591*	*0.0707*	3
Commodities						
1	Cocoa	625	*0.4200*	*0.6553*	*0.3852*	3
2	Coffee (Arabica)	625	*0.0575*	*0.1086*	*0.1003*	3
FX						
USDXXX						
1	USDDKK	625	*0.0534*	0.0029	0.0200	1
2	USDEUR	625	*0.0509*	0.0019	0.0160	1
3	USDKRW	625	*0.0717*	*0.1064*	*0.1078*	3
4	USDKRW	1250	*0.2834*	*0.2250*	*0.2573*	3
5	USDNAD	625	*0.2116*	*0.2082*	*0.2136*	3
6	USDNOK	625	*0.1447*	*0.2713*	*0.1329*	3
7	USDNZD	625	0.0375	*0.0895*	*0.0699*	2
8	USDSDG	1250	*0.4366*	*0.4992*	*0.4222*	3
EURXXX						
1	EURBGN	625	*0.1229*	0.0016	*0.0531*	2
2	EUREGP	625	0.0027	*0.1147*	0.0182	1
3	EURHUF	625	*0.0810*	*0.0857*	*0.1221*	3
4	EURIDR	625	0.0486	0.0117	*0.0911*	1
5	EURNAD	625	0.0153	*0.0980*	0.0311	1
6	EURNOK	625	0.0281	*0.0781*	*0.0545*	2
7	EURUAH	625	*0.1579*	0.0385	*0.1941*	2
8	EURZAR	625	0.0173	*0.0606*	0.0359	1
Other currency pairs						
1	AUDCHF	625	*0.0643*	*0.1595*	*0.0691*	3
2	CADCHF	625	*0.4070*	*0.1245*	*0.3713*	3

Source: own study.

4.2. Change of the Hurst coefficient over time

The obtained results are presented in Table 2 and Figures 1–2.

4.2.1. Comparison of the Hurst exponents at the end and the beginning of the time interval

For the 1250-session window:

Comparison of the last Hurst exponent (i.e., calculated after the session on 31.12.2019 and designated as $H_{31122019}$) with the first available for a given financial instrument (H_0) leads to the conclusion that for all the groups of financial instruments, except for Equities listed on the WSE, the percentage of cases when the condition $H_0 < H_{31122019}$ was met is lower than 10%, and for Equities listed on the WSE it was equal to 31.05%. In the case of the Currency market, the lowest percentage value was recorded in the segment of Other currency pairs and was equal to 0.00%. The percentage of cases in the Commodity group (9.76%) was higher than in the Equity Indexes group (4.05%).

In the group of Equity indices, the value of the H_0 exponent was higher than $H_{31122019}$ for three indices: BUENOS, NIFTY50, NASDAQ 100, in turn for the Commodity market for the following commodities: heating oil, lumber, natural gas, and oats. On the Currency market, this condition was met for the following currency pairs: USDEGP, USDXPD, EUREGP, EURHRK, EURXDR, JPYEGP, and thus in three cases when EGP was the second currency.

Based on the obtained results, it can be concluded that in the longer time horizons the value of the H exponent decreased in all the analysed capital market segments. That means that the value of the fractal dimension and thus the level of investment risk increased. The limitation of the conclusion formulated in this way are the different lengths of analysed time horizons, for each of the examined financial instruments. This disadvantage is deprived of the analysis in which the Hurst exponent values measured in the same dates are confronted with each other. The first date is the session on 31.12.2019 and the second – the date 250 sessions earlier (i.e., $H_{31122019}$ vs. H_{-1250}).

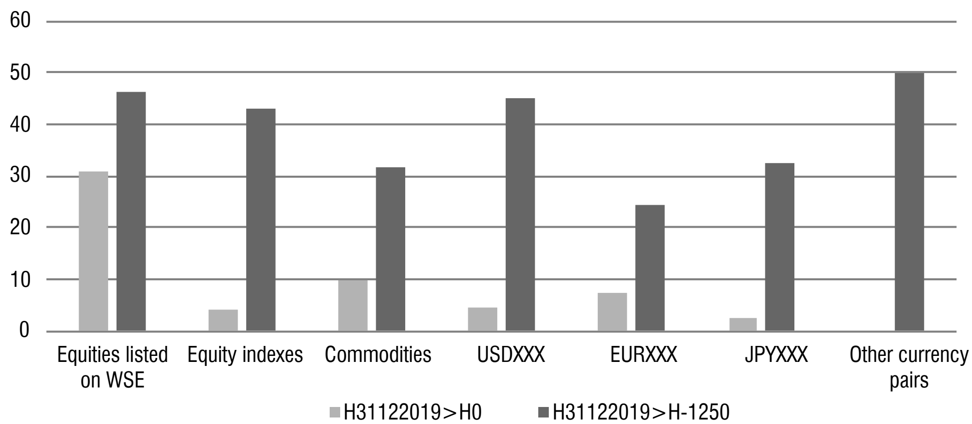
The percentage of cases when the value of the Hurst exponent after the session falling on 31.12.2019 was higher than 1250 sessions earlier (i.e. $H_{31122019} > H_{-1250}$), proved to be higher in all the analysed groups of assets than the percentage of cases meeting the condition $H_{31122019} > H_0$.

The percentages of cases meeting the equation $H_{31122019} > H_{-1250}$ ranged from 24.39% (EURXXX) to 40% (Other currency pairs). Also, it can be concluded that in the approx. five-year horizon the value of the H exponent decreased (which means that the value of the fractal dimension increased) in all the analysed capital market segments, except for the Other currency pairs segment, but this trend was not so visible as for the condition: $H_{31122019} > H_0$.

In the group of Equity indexes, the value of the H_0 was higher than the H_{-1250} for the following indices: All Ordinaries, AMEX, ATX, BET, Bovespa, BSE Sensex, BSHARES, BUX, China A50, DJIA, Eurostoxx 50, FTSE 100, Hang Seng, HEX, Kospi, Merval, MOEX, Nifty 50, NZ 50, OMX Riga, OMX Talinn, PX, RTSI, S&P ASX 200, Shanghai Composite, Straits Times, Taiex, TECDAX, TSE300, UK 100, UX, WIG. In the group of Commodities for the following commodities: Brent, CO_2, copper, frozen orange juice, gas oil, lumber, palladium, platinum, sugar, WTI. On the Currency market, this condition was met for the following currency pairs:

a) In the USDXXX segment: USDBGN, USDBRL, USDCAD, USDCNY, USDCZK, USDEURO, USDGBP, USDHKD, USDHRK, USDIDR, USDISK, USDMYR, USDNOK, USDPLN, USDRUB, USDSEK, USDTHB, USDTRY, USDXDR,
b) In the EURXXX segment: EURBGN, EURCZK, EUREGP, EURGBP, EURMXN, EURNOK, EURNZD, EURRUB, EURTRY, EURXPT,
c) In the JPYXXX segment: JPYIDR, JPYCHF, JPYGBP, JPYNOK, JPYPLN, JPYRUB, JPYSEK, JPYTRY, JPYUAH, JPYBRL, JPYCAD, JPYEGP, JPYXPT,
d) In the Other currency pairs segment: AUDCHF, GBPAUD, GBPCAD, GBPPLN, GBPNZD, CHFPLN.

Figure 1. Comparison of Hurst coefficients H_0, $H_{31122019}$ and H_{-1250}, (H = 1250), (%)



Source: own study.

Both conditions, i.e. $H_{31122019}> H_0$ and $H_{31122019}> H_{-1250}$ were met simultaneously for the following instruments:

a) Equities listed on the WSE: 08N (Octavia), Action, AMRest, Arcus, Assecobs, Atrem, Bogdanka BOS, Bumech, CDProjekt, CEZ, Cormay, Decora, Elbudowa, Elkop, Emcinsmed, Energoins, Eurotel, Handlowy, Introl, Kania, Krakchemia, Kredyt Inkaso, Lotos, MOL, MWTrade, Nowa Gala, Oponeo, Orange, Orbis, Panova, Pozbud, Pragmafa, Projprzem, Puławy, Radpol, Skyline, Śnieżka, Sygnity, Talex, Unibep, Ursus, Windexus, Wadex, Wojas – i.e., for 14.06% of analysed companies.
b) Equity indices: NIFTY50 – i.e., for 1.35% of the analysed indices.
c) Commodities: heating oil and lumber, i.e., for 4.88% of the analysed commodities.

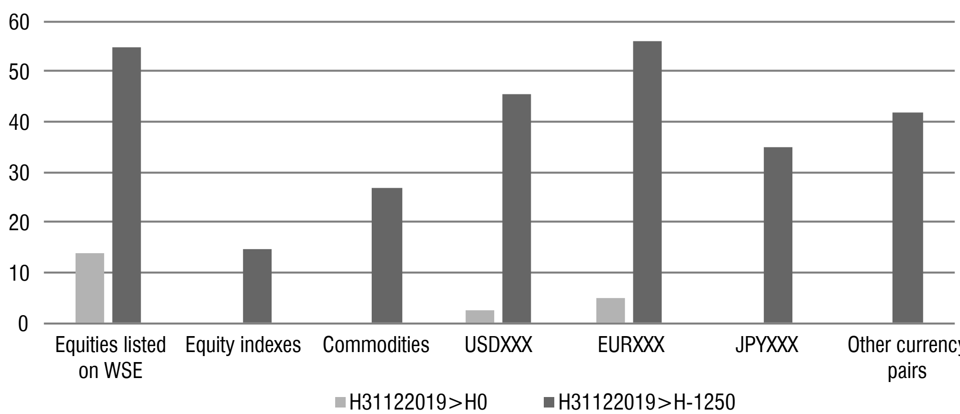
d) Currency market EURXXX segment: EUREGP, i.e., for 2.43% of exchange rates in the EURXXX segment.
e) Currency market JPYXXX segment: JPYEGP, i.e., for 2.50% of exchange rates in the JPYXXX segment.

For the 625-session window:

Combining the Hurst exponents, calculated after the session on 31.12.2019 ($H_{31122019}$) with the first available for a given financial instrument (H_0), the percentage of cases when the condition $H_{31122019} > H_0$ is met, resulted to be lower less than 15% for all the analysed groups of assets, with 0% for Equity indices, Commodities and Other currency pairs. The highest percentage was recorded for the segment of Equities listed on the WSE and equal to 13.75%.

Similarly to the 1250-session horizon, also in the case of the 625-session horizon, the value of the *H* exponent decreased (and thus the value of the fractal dimension increased) in all the analysed groups of assets. However, the percentage of cases where the equation $H_{31122019} > H_0$ is met is smaller for the 625-session horizon. This relationship is true for all the analysed groups of financial instruments. In turn, the percentage of conditions when the equation $H_{31122019} > H_{-1250}$ is met was higher than 50% in two asset groups: EURXXX (56.10%) and Equities listed on the WSE (54.75%). In the case of other groups of assets, this percentage was lower than 50%, while its lowest value was recorded in the case of Equity Indices (14.86%). It can, therefore, be concluded that over the past five years (around 1250 sessions) the value of the Hurst exponent decreased in the following asset groups: Equity indices, Commodities, JPYXXX, Other currency pairs, and USDXXX. For the last two segments, the percentage of cases when the condition $H_{31122019} > H_{-1250}$ was met was equal to 41.67% and 45.24%, respectively. In the group of Equities listed on the WSE and EURXXX, the percentage of cases with an increase in the value of the Hurst exponent was higher than 50%.

Figure 2. Comparison of Hurst coefficients H_0, $H_{31122019}$ and H_{-1250}, (H=625), (%)



Source: own study.

Table 2. Analysis of changes of the H exponents over time for 625 and 1250 session windows (%)

H:	1250		625		1250 (Full time window)			625 (Full time window)			1250 (1250-session window)			625 (625-session window)		
	Condition: $H_{31122019} > H_0$	Condition: $H_{31122019} > H_{-1250}$	Condition $H_{31122019} > H_0$	Condition $H_{31122019} > H_{-1250}$	Parameter m>0	Parameter a>0	$R^2>0.6$	Parameter m>0	Parameter a>0	$R^2>0.6$	Parameter m>0	Parameter a>0	$R^2>0.6$	Parameter m>0	Parameter a>0	$R^2>0.6$
Equities listed on WSE	31.05	46.19	13.75	54.75	29.41	100.00	32.03	8.13	100.00	24.69	40.99	93.69	35.59	41.94	94.27	17.20
Equity indexes	4.05	43.24	0.00	14.86	10.81	100.00	29.73	1.35	100.00	27.03	47.22	91.67	30.56	47.30	87.84	9.46
Commodities	9.76	31.71	0.00	26.83	21.95	100.00	7.32	2.44	100.00	17.07	34.15	92.68	19.51	17.07	92.68	29.27
USDXXX	4.76	45.24	2.38	45.24	4.76	100.00	47.62	2.38	100.00	42.86	33.33	95.24	21.43	11.90	95.24	23.81
EURXXX	7.32	24.39	4.88	56.10	12.20	100.00	31.71	7.32	100.00	26.83	26.83	100.00	29.27	24.39	97.56	17.07
JPYXXX	2.50	32.50	0.00	35.00	7.50	100.00	35.00	0.00	100.00	20.00	12.50	95.00	52.50	12.50	97.50	27.50
Other Currency Pairs	0.00	50.00	0.00	41.67	0.00	100.00	50.00	0.00	100.00	50.00	16.67	100.00	33.33	16.67	100.00	16.67

Source: own study.

Both conditions, i.e., $H_{3122019} > H_0$ and $H_{31122019} > H_{-1250}$ were met simultaneously in the group of Equities listed on the WSE: Apsenergy, Braster, Bumech, CEZ, CNT, Elbudowa, Elemental, Emcinsmed, Euco, GLCosmed, Groclin, Grodno, Introl, IMTrade, Izoblok Kania, Kredyt Inkaso, Lena, MediaCap, MWTrade, Nowagala, OpenFin, Otlog, Panowa, Skarbiec, Śnieżka, Sonel, Tauronpe, Torpol, Trakcja, Unibep, Unimot, Yolo, Zepak, ZPUE – i.e., for 11.88% of all the analysed companies.

In turn, both conditions, i.e., $H_{3122019} > H_0$ and $H_{31122019} > H_{-1250}$ were met for the 1250- as well as for 625-session horizon only for the following companies listed on the WSE: Bumech, CEZ, Elbudowa, EMCINSMED, Introl, Kania, Kredyt Inkaso, MWTRADE, Nowa Gala, Śnieżka, Unibep, i.e., for 3.44% of the analysed companies.

4.2.2. Analysis of the value of the m parameter in the regression model

For the 1250-session window:

Analysis of the values of *m* parameters obtained in the linear regression model for all the *H* values, such that $H \in (H_0, H_{31122019})$, when the condition $m > 0$ was met, indicates that in all the groups of the analysed financial instruments the percentage was lower than 30%, with the highest in the group of Equities listed on the WSE (29.41%), and the lowest in the group of Other currency pairs (0%). In the second place, in terms of the highest percentage of positive values of the *m* parameter, there were reported Commodities (21.95%), and in the third the Forex market with its segment of EURXXX (12.20%). The percentage of R^2, exceeding the value of 0.6 was the highest in the segment of Other currency pairs (50%), before the USDXXX (47.62%) and JPYXXX (35%). The lowest percentage of R^2, greater than 0.6, was registered for Commodities (7.32%). The obtained results allow one to draw the conclusion that in most cases the value of the *m* parameter was lower than zero ($m < 0$) – see Figures 3–6.

In turn, in the linear regression model for all the *H* values, such that $H \in (H_{-1250}, H_{31122019})$, the highest percentage of $m > 0$ was observed in the group of Commodities (47.22%), ahead of the Equities listed on the WSE (40.99%), while the lowest in the JPYXXX segment (12.50%). The percentage of cases when the value of *m* was positive was higher for $H \in (H_{-1250}, H_{31122019})$ than for $H \in (H_0, H_{31122019})$. This rule applies to all the analysed groups of assets. The difference was in the analysed percentage when $m > 0$ was the highest in the following groups: Commodities (36.41 percentage points) and USDXXX (28.57 percentage points). The percentage of R^2, exceeding 0.6, was the highest in the JPYXXX segment (52.50%), which was ahead of the group of Equities listed on the WSE (35.59%) and Other currency pairs (33.33%). The lowest percentage of R^2, greater than 0.6, was the share of the Commodity group (19.51%).

The value of the *m* parameter was positive (m>0) at the same time for two time horizons ($(H_{-1250}, H_{31122019})$ and $(H_0, H_{31122019})$) for the following groups of assets:

a) Equities listed on the WSE: Action, AMRest, Arcus, Arteria, Assecobs, Bogdanka, BOS, Bumech, Cormay, Emcinsmed, Energoins, Eurotel, IFFIRMA, Introl, Invista, Kania

Kernel, Krakchemia, Kredyt Inkaso, LSISOFT, Mercor, MWTrade, Oponeo, Orco Group, Panova, PEP, Pozbud, Pragmafa, Pragmaink, Radpol, Śnieżka, Sygnity, Unibep, Vindexus, Wadex, Wielton, Wojas, i.e., for 18.09% of the analysed companies for which both H_{1250} *and* H_{625} were determined.

b) Equity indices: Amex, NZ50, SASESLCT, SHANGHAI COMPOSITE, i.e., for 5.55% of the analysed indexes.
c) Commodities: BRENT, heating oil, natural gas, oil, oats, and ULSD NY HARBOR, i.e., for 14.63% of assets in this group.
d) Currency market EURXXX segment: EURBGN, EURISK, EURXDR, i.e., for 7.32% of exchange rates in the EURXXX segment.
e) Currency market JPYXXX segment: JPYEGP, i.e., for 2.50% of exchange rates in the JPYXXX segment.

For the 625-session window:

Analysis of the *m* parameter obtained in the linear regression model for all the *H* values, i.e., for the range: $H \in (H_0, H_{31122019})$, proved that in all the analysed groups of financial instruments the percentage of $m>0$ was lower than 9%. The highest percentage was observed in the group of Equities listed on the WSE (8.13%), and the lowest in the Other currency pairs and JPYXXX segment (equal to 0% in both cases). In terms of the highest percentage of positive values of the *m* parameter, in the second place the segment of EURXXX (7.32%) was classified and in the third place the group of Commodities (2.44%). The highest percentage of R^2, exceeding 0.6, was registered for the segment of Other currency pairs (50%), which ranked before the segment of USDXXX (42.86%) and the Equity indexes group (27.03%). The lowest value of R^2 percentage greater than 0.6 was registered for the Commodity group (17.07%).

On the other hand, in the linear regression model constructed for all the *H* values, such that $H \in (H_{-1250}, H_{31122019})$, the highest percentage of $m>0$ was observed for the Commodities (47.30%), ahead of the Equities listed on the WSE (41.94%), while the lowest in the USDXXX segment (11.90%). The percentage of cases when the value of *m* was positive proved to be higher for the $H \in (H_{-1250}, H_{31122019})$ than for $H \in (H_0, H_{16082019})$, for all the analysed groups of assets. The highest difference was observed for the Equity indexes group (45.95 percentage points) and Equities listed on the WSE (33.81 percentage points). The percentage of R^2, exceeding 0.6, was the highest in the segment of Commodities (29.27%), which was ahead of the JPYXXX (27.50%) and USDXXX (23.81%) segments. The lowest percentage of R^2 greater than 0.6, was observed for Equity indexes group (9.46%).

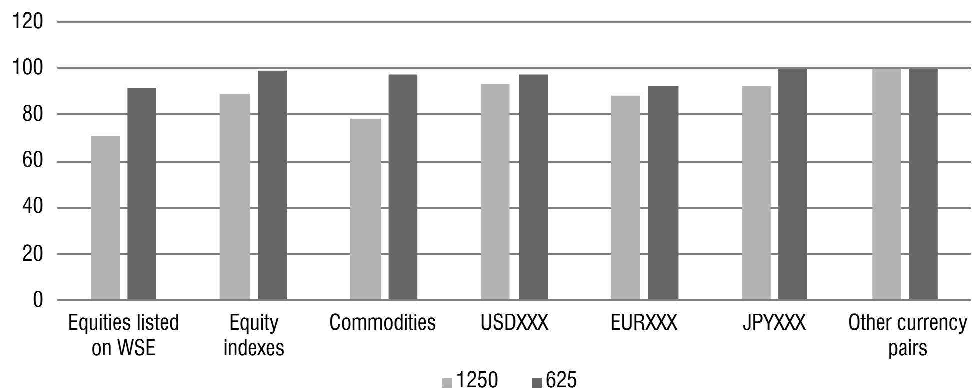
The value of the *m* parameter was positive ($m>0$) at the same time for two time horizons ($(H_{-1250}, H_{31122019})$ and $(H_0, H_{31122019})$) for the following groups of assets:

a) Equities listed on the WSE: Action, CEZ, Energoins, Introl, Kredyt Inkaso, Pragmafa, Quantum, Radpol, Rafamet, Tauronpe, i.e., for 3.58% of the analysed companies for which both H_{1250} *and* H_{625} were determined.

b) The currency market EURXXX segment: EURISK, i.e., for 2.43% of exchange rates in the EURXXX segment.

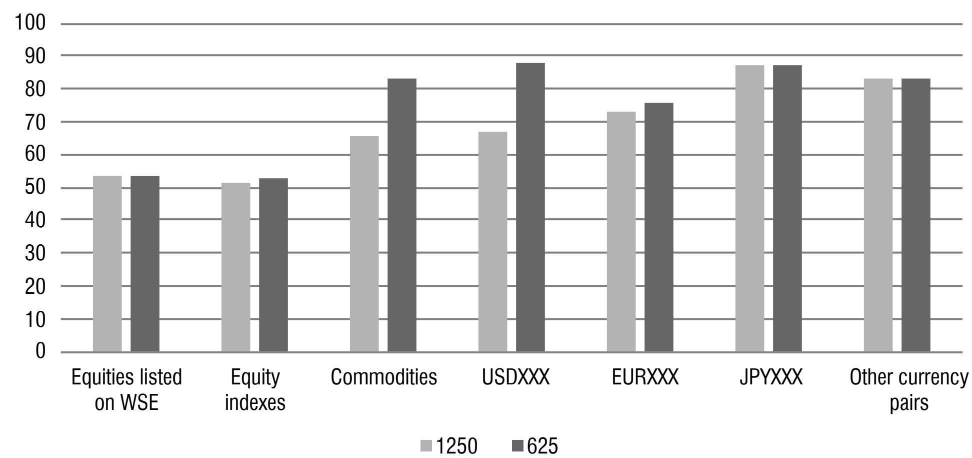
The value of the *m* parameter was positive (m>0) at the same time for two time horizons ($(H_{-1250}, H_{31122019})$ and $(H_0, H_{31122019})$) and for two time windows, i.e., 1250- and 625-sessions, only in the segment of Equities listed on the WSE: Action, Energoins, Introl, Kredyt Inkaso, Pragmafa, Radpol, i.e., for 2.86% of the analysed companies for which both H_{1250} and H_{625} were determined.

Figure 3. Percentage of cases in the analysed asset groups when m < 0 (all the sessions included in the model, %)



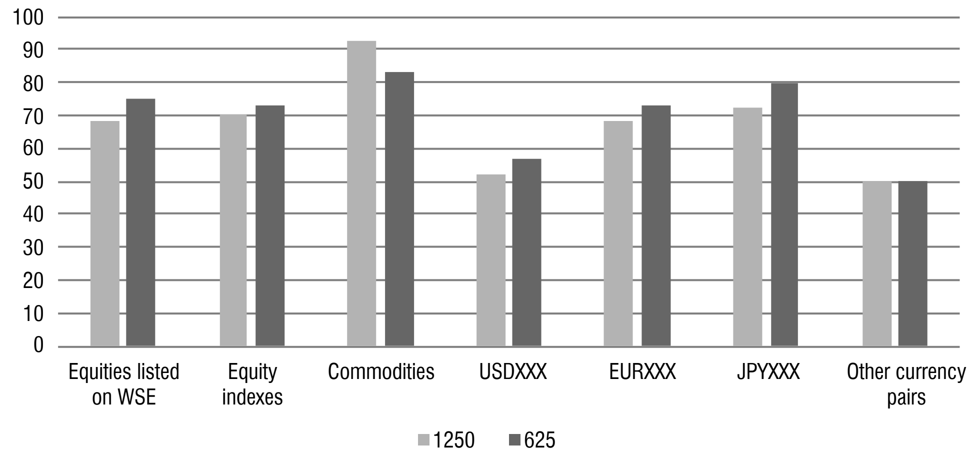
Source: own study.

Figure 4. Percentage of cases in the analysed asset groups when m < 0 (only sessions from the last 5 years preceding 31.12.2019 included in the model, %)



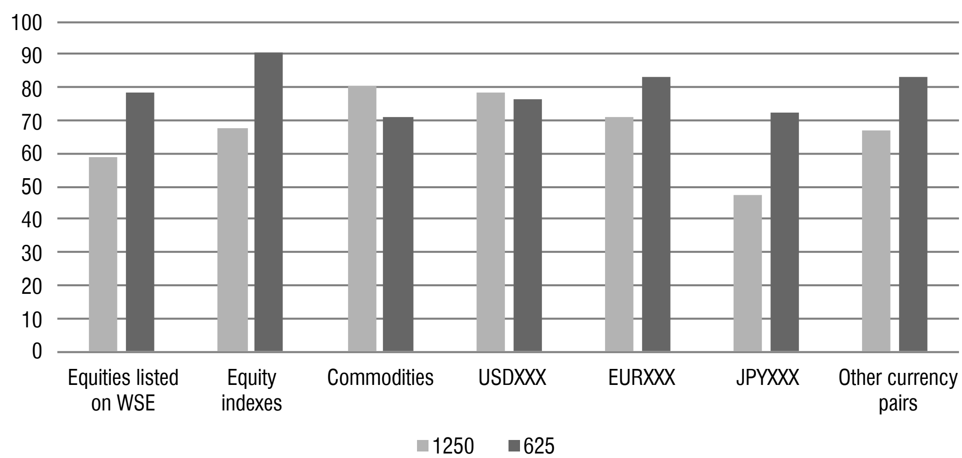
Source: own study.

Figure 5. Percentage of cases in the analysed asset groups when $R^2 > 0.6$ (all the sessions included in the model, %)



Source: own study.

Figure 6. Percentage of cases in the analysed asset groups when R2 > 0.6 (only sessions from the last 5 years preceding 31.12.2019 included in the model, %)



Source: own study.

4.2.3. Analysis of cases when conditions described in 4.2.1 and 4.2.2. are fulfilled at the same time

I) This chapter will discuss cases such that both conditions are met regarding the relationship of the *H* exponents and the positive sign of the *m* coefficient, i.e., $H_{31122019} > H_0$ and m>0, when the *m* parameter was calculated for the investment horizon $(H_0, H_{31122019})$.

II) $H_{31122019} > H_{-1250}$ and m>0, when the *m* parameter was calculated for the investment horizon $(H_{-1250}, H_{31122019})$.

For the 1250-session window:

Condition I was met in the following groups of assets:

a) Equities listed on the WSE: 71 (23.20%), for which the *H* parameter was calculated;
b) Equity indices: 2 cases, i.e., BUENOS and NIFTY50, i.e., for 2.70% of all the analysed Equity indices;
c) Commodities: heating oil, natural gas, oil, oats, i.e., for 9.76% of all the analysed Commodities;
d) Currency market, USDXXX segment: USDEGP, i.e., for 2.38% of exchange rates in the USDXXX segment;
e) Currency market, EURXXX segment: EURHRK, EURXDR, i.e., for 4.88% of exchange rates in the EURXXX segment;
f) Currency market, JPYXXX segment: JPYEGP, i.e., for 2.50% of exchange rates in the JPYXXX segment.

Condition II, in turn, was met in the following asset groups:

a) Equities listed on the WSE: 110 (36.67%), for which the value of the *H* parameter was calculated;
b) Equity indexes: AMEX, ATX, BOVESPA, BSHARES, BSE SENSEX, BUX, DJIA, FTSE 100, HANG SENG, IPSA, KOSPI, MOEX, NIFTY50, NZ50, OMX RIGA, OMX TALLINN, PX, SHANGHAI COMPOSES, STRAITS TIMES TAIEX, TECDAX, TSE300, UK100, WIG, i.e., 33.33% for which the value of the *H* parameter was obtained;
c) Commodities: heating oil, gas oil, natural gas, oats, i.e., for 9.76% of all the analysed raw materials;
d) Currency market, USDXXX segment: USDEGP, i.e., for 2.38% of exchange rates in the USDXXX segment;
e) Currency market, EURXXX segment: EURHRK, EURXDR, i.e., for 4.88% of exchange rates in the EURXXX segment;
f) Currency market, JPYXXX segment: JPYEGP, i.e., for 2.50% of exchange rates in the JPYXXX segment.

For the 625-session window:

Condition I was met in the following groups of assets:

a) Equities listed on the WSE: 15 (4.69%), for which the *H* parameter was calculated;
b) Currency market, USDXXX segment: USDEGP, i.e., for 2.38% of exchange rates in the USDXXX segment;
c) Currency market, EURXXX segment: EURHRK, EURXDR, i.e., for 4.88% of exchange rates in the EURXXX segment.

In turn, condition II was met in the following groups:

a) Equities listed on the WSE: 66 (24.72%), for which the value of the *H* parameter was calculated;

b) Equity indexes: IPC, MEXICIP, MOEX, S&P ASX 200, i.e., 5.41% for all the analysed indices;
c) Commodities: coffee (Robusta), frozen orange juice, sugar, i.e., for 7.31% of all the analysed commodities;
d) Currency market, USDXXX segment: USDIDR, USDMYR, i.e., for 4.76% of exchange rates in the USDXXX segment;
e) Currency market, EURXXX segment: EURBRL, EURHUF, EURNZD, EURRON, EURXPT, i.e., for 14.63% of exchange rates in the EURXXX segment;
f) Currency market, JPYXXX segment: JPYIDR, i.e., for 2.50% of exchange rates in the JPYXXX segment;
g) Currency market, Other currency pairs segment: AUDPLN, i.e., for 8.33% of exchange rates in the Other currency pairs segment.

4.2.4. Analysis of cases when four conditions are met at the same time

This chapter will reveal cases when four conditions are met at the same time regarding the relationship of the *H* parameters and the positivity of the *m* coefficient, i.e.:

$$\begin{cases} H_{31122019} > H_0 \text{ and } m > 0 \quad \text{for } m \in (H_0, H_{31122019}) \\ \text{and} \\ H_{31122019} > H_{-1250} \text{ and } m > 0 \text{ for } m \in (H_{-1250}, H_{31122019}) \end{cases}$$

For the 1250-session window:

All four conditions were met only in the group of Equities listed on the WSE: 28 examples (13.33%), for which the value of the *H* parameter was calculated.

For the 625-session window:

All four conditions were met only in the group of Equities listed on the WSE: 4 (CEZ, Introl, Kredyt Inkaso, Tauronpe), i.e., for 1.50% financial instruments for which the value of the *H* parameter was calculated.

4.3. Normal distribution of the average values of the Hurst exponents: H_{1250} and H_{625}

The results obtained are presented in Table 3.

The distribution of the average values of the Hurst exponents is a normal distribution except for the following groups of assets:
a) Commodities for H_{1250}
b) USDXXX segment for H_{1250}

c) USDXXX segment for H_{625}
d) JPYXXX segment for H_{625}

In all four cases, the p-value was lower than 0.05 for all three tests (JB, SW and DA). In turn, in the case of the Equity indexes and H_{625}, the null hypothesis was rejected with the use of SW test, when the p-value was equal to 0.0499, while in the case of the remaining JB and SW tests, p-value, was higher than 0.05 and amounted to: 0.2079 and 0.1970, respectively.

Table 3. The p-value parameters calculated with the use of normal distribution tests (JB, SW, and DA) for average values of H_{1260} and H_{625} (p-value<0.05 in italics)

	H_{1250}			H_{625}		
	JB	SW	DA	JB	SW	DA
Equities listed on the WSE	0.6646	0.6600	0.6811	0.6843	0.4095	0.6283
Equity indexes	0.2062	0.0730	0.1904	0.2079	*0.0499*	0.1970
Commodities	*0.0000*	*0.0001*	*0.0000*	0.0674	0.0757	0.0267
USDXXX	*0.0000*	*0.0000*	*0.0000*	*0.0000*	*0.0001*	*0.0000*
JPYXXX	0.1543	0.4084	0.0638	*0.0305*	*0.0227*	*0.0066*
Other currency pairs	0.6583	0.7737	0.2743	0.8339	0.8772	0.7660

Source: own study.

The comparison of the results obtained for the normal distribution of returns (4.1) and normal distribution of the average Hurst exponents (4.3) leads to the following conclusion: financial instruments for which there were no reasons to reject the null hypothesis both when testing the normal distribution of logarithmic returns and the normal distribution of 625-sessions average values of the Hurst exponents are:

a) Equities listed on WSE: KGHM, MBANK, PKO BP;
b) Equity indexes: PSEI, WIG;
c) EURXXX: EURBGN, EUREGP, EURHUF, EURIDR, EURNAD, EURNOK, EURUAH, EURZAR;
d) Other currency pairs: AUDCHF, CADCHF.

In the case of other financial instruments, the null hypothesis of the normality of the distribution of returns or the normality of the distribution of the average 625- and 1250-session Hurst exponents were rejected.

Summary

The main conclusions resulting from the study may be formulated as follows:

a) For the majority of the analysed financial instruments, the distribution of logarithmic rates of return is not a normal distribution. The highest percentage when rates of return were normally distributed were recorded in the Foreign exchange market segment and the lowest in the segment of Equities listed on the WSE. In the case of a 625-session window, the number of cases when there was no reason to reject the null hypothesis was definitely higher than for a 1250-session window. Therefore, in shorter time horizons, the distribution of return rates is closer to the normal distribution than in the longer term.

b) The value of the *H* exponents decreased in all the analysed time windows, which means an increase in the fractal dimension (*d*), and thus the investment risk in financial markets.
c) In most cases the values of the *m* parameter, obtained in the linear regression model for all the *H* values, were lower than zero ($m<0$).
d) The values of $m>0$ were achieved for less than 30% of the analysed instruments, with the highest percentage in the group of Equities listed on the WSE (29.41%), followed by Commodities (21.95%), and the Forex market with its segment of EURXXX (12.20%).
e) With few exceptions, the distribution of average values of the Hurst exponents is a normal distribution.

The decrease in the value of the Hurst exponent, demonstrated during the analysis, and thus the increase in the value of the fractal dimension, which means an increase in investment risk is consistent with the conclusions made by Przekota [2012, pp. 186–187], who proved that in the case of defining risk as volatility and assuming that greater volatility means greater risk, then lower values of the fractal dimension mean lower risk, and higher values of the fractal dimension – higher risk.

The highest values of the fractal dimension were recorded by Orzeszko, who, using the segment-variation method and the segment length selection procedure developed by Zwolankowska [2002, pp. 209–224], proved that in the period from 02.01.2001 to 03.03.2009 the highest fractal dimension belonged to the WIG20 index (1.4360), before the following indexes: WIG (1.3580), mWIG40 (1.3090) and sWIG80 (1.2800). In turn, Buła demonstrated that the medians of estimated fractal dimensions for capital market indices (WIG20, mWIG40 and sWIG80) met the relationship [Buła, 2019, p. 109]:

$$ME_{WIG20} > Me_{mWIG40} > Me_{sWIG80}$$

The problem of the relationship between the level of investment risk and the level of the fractal dimension is relatively rarely raised. While attention is paid to the occurrence of such dependence, it is stated that such a relation is one-way, according to which the increase in the fractal dimension is equated with the increase in the level of investment risk, and its decrease – with the decrease in investment risk. The justification is based on the use of the notion of a range of cumulative logarithmic rate of return (or exchange rate) for a given time series [Zwolankowska, 2000, p. 268; Waściński, Przekota, 2012, pp. 81–82]. According to Buła, the fractal dimension can be used as a complementary measure to classic risk measures, because the fractal dimension and classic measures provide the investor with non-duplicate information [Buła, 2019, pp. 115–117]. Similar conclusions can be drawn from Zwolankowska's work [2001, p. 113].

The presented results are in line with those published in relation to the S&P 500 index by Alvarez-Ramirez et al. [2008, pp. 6159–6169] and Dominique and Rivera [2011, pp. 1–6], and in the case of the NASDAQ index by Glenn [2007]. The results also confirm the conclusions obtained by Huang and Yang [1999] in the process of analysing the Hurst exponents for the indices: DJIA, NASDAQ and S&P 500, as well as results revealed by Henry [2002, pp. 725–729] for DAX, TOPIX and Nikkei 225.

On the other hand, the obtained results are in contradiction to the conclusions published by: Onali and Goddard [2010] for the FTSEMIBTEL and PX indices, by Henry [2002, pp. 725–729] for the Kospi index, Berg and Lyhagen [1996] for OMX Stockholm Index, by Barkoulas et al. [1996, pp. 253–259] for three indices: DJIA, NASDAQ and S&P 500 and by Cajueiro and Tabak [2004, pp. 349–352] for the MEXIPC and IPC indexes. In the case of the S&P 500 index, the obtained results contradict the conclusions drawn from the work of Granger and Ding [1995, pp. 67–91], Granger and Hyung [2004, pp. 399–421], as well as Lo [2004, pp. 15–29] and Jacobsen [1995, pp. 37–52].

The conclusions, regarding the normality of the distribution of average *H* values are consistent with those published by Kale and Butar [2011, pp. 8–19]. The obtained results for WIG index confirm the Kowager's conclusions for H_{1250}=0.528926>0.5 but are different for H_{625} (= 0.427175) <0.5. In the case of mBank the results H_{1250}=0.44226 and H_{625}=0.41627, e.g., lower than 0.5, are different from those included in the Kowager's work, where *H* was higher than 0.5 [2009, pp. 157–167]. However, it is worth noting that the longest investment horizon taken into account by Kowager, i.e., 42 weeks (= 294 days) is more than twice and four times shorter than the horizon of 625 and 1250 sessions applied in this analysis.

The results presented in the research partly confirm Rzeszótko's observations [2016, pp. 131–141]. Values H_{1250}>0.5 were obtained for equity prices of following banks: BOŚ (0.621671), Getin (0.585317), Getin Noble (0.631989), Handlowy (0.518972), and values meeting condition H_{1250} <0.5 for Alior (0.478313), ING BS (0.486699), mBANK (0.442588), Pekao SA (0.489334), PKO BP (0.481842) and Santander PL (0.495240). The results of this study are to some extent consistent with those presented by Buła and regarding the persistence of time series of metal prices, in particular the prices of palladium, zinc, and nickel [Buła, 2015]. In the case of other commodities, the results presented by Buła were not confirmed in the research. However, it should be remembered that weekly (rather than daily) logarithmic rates of return were taken into account in Buła's study.

This article may be considered as a voice in the ongoing discussion on the effectiveness of financial markets.

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Annex

The list of analysed financial assets and the dates of their first listing disclosed in the database (portals: stooq.pl and bossa.pl)

No.	Equities listed on WSE	First quotation	No. (Cont.)	Equities listed on WSE	First quotation	No. (Cont.)	Equity indexes	First quotation	No. (Cont.)	FX market	First quotation
1	06N (MAGNA POLONIA)	12.06.1997	161	LARQ	20.06.2006	321	AEX	04.01.1983		**EURXXX**	
2	08N (OCTAVIA)	12.06.1997	162	LCCORP	29.06.2007	322	ALL ORDINARIES	31.10.1989	478	EURAUD	02.01.1980
3	11BIT	28.10.2010	163	LENA	01.06.2005	323	AMEX	03.01.1995	479	EURBGN	23.10.1995
4	4FunMedia	30.11.2010	164	LENTEX	08.05.1997	324	ATHEX COMPOSITE	02.01.1987	480	EURBRL	03.01.1995
5	ABPL	21.09.2006	165	LIBET	28.04.2011	325	ATX	11.11.1992	481	EURCAD	04.01.1971
6	ACAUTOGAZ	11.08.2011	166	LOTOS	09.06.2005	326	BEL20	29.10.1991	482	EURCHF	04.01.1971
7	Action	24.07.2006	167	LPP	16.05.2001	327	BET	31.10.2000	483	EURCLP	25.10.1993
8	Agora	20.04.1999	168	LSISOFT	21.12.2006	328	BIST 100	03.01.1988	484	EURCNY	02.01.1991
9	Agroton	08.11.2010	169	LUBAWA	18.11.1996	329	BOVESPA	02.01.1995	485	EURCZK	23.10.1995
10	Ailleron	12.10.2011	170	MABION	10.08.2010	330	BSE SENSEX	03.04.1979	486	EURDKK	02.01.1991
11	Alchemia	19.05.1998	171	MAKARONPL	18.04.2007	331	BSHARES	11.05.1998	487	EUREGP	16.11.1995
12	Alior	14.12.2012	172	MANGATA	04.03.2005	332	BUENOS	08.10.1996	488	EURGBP	04.01.1971
13	Alta	20.01.1999	173	MARVIPOL	19.06.2008	333	BUX	02.01.1991	489	EURHKD	02.01.1991
14	Alumetal	17.07.2014	174	MBANK	06.10.1992	334	CAC40	02.01.1969	490	EURHRK	23.10.1995
15	Ambra	22.06.2005	175	MCI	01.02.2001	335	China A50	19.05.2010	491	EURHUF	23.10.1995
16	Amica	08.09.1997	176	MDIENERGIA	12.06.1997	336	CSE ALL SHARES	13.07.2000	492	EURIDR	25.10.1993
17	AMREST	27.04.2005	177	MEDIACAP	24.07.2008	337	DAX	28.09.1959	493	EURILS	25.10.1993
18	Apator	24.04.1997	178	MEDICALG	09.11.2011	338	DJIA	27.05.1896	494	EURINR	02.01.1991
19	Aplisens	25.05.2009	179	MENNICA	02.04.1998	339	DJTA	26.10.1896	495	EURISK	23.10.1995
20	Apsenergy	08.08.2013	180	MERCOR	19.07.2007	340	DJUA	02.01.1929	496	EURJPY	04.01.1971
21	Arctic	23.10.2009	181	MEXPOLSKA	25.05.2012	341	EOE	02.01.1995	497	EURKRW	13.04.1981
22	Arcus	10.09.2007	182	MILLENNIUM	13.08.1992	342	EURO STOXX 50	15.08.2011	498	EURMXN	02.01.1991
23	Arteria	15.12.2006	183	MIRACULUM	13.02.2007	343	FTSE 100	22.10.1992	499	EURMYR	02.01.1991
24	ASBIS	30.10.2007	184	MIRBUD	29.12.2008	344	FTSE MIBTEL	02.01.1998	500	EURNAD	02.01.1991

No.	Equities listed on WSE	First quotation	No. (Cont.)	Equities listed on WSE	First quotation	No. (Cont.)	Equity indexes	First quotation	No. (Cont.)	FX market	First quotation
25	Assecobs	19.11.2007	185	MLPGROUP	28.10.2013	345	FTSE250	31.12.1985	501	EURNOK	02.01.1991
26	Assecopol	02.06.1998	186	MOJ	13.06.2007	346	HANG SENG	24.11.1969	502	EURNZD	02.01.1991
27	Assecosee	28.10.2009	187	MOL	11.12.1995	347	HEX	02.01.1995	503	EURPHP	25.10.1993
28	Astarta	17.08.2006	188	MONNARI	20.12.2006	348	IBEX 35	05.01.1987	504	EURPLN	16.05.1991
29	Atende	28.05.2012	189	MORIZON	08.04.2011	349	IDX COMPOSITE	09.04.1990	505	EURRON	23.10.1995
30	Atlantapl	10.01.2005	190	MOSTALPLC	15.09.1998	350	IPC	08.11.1991	506	EURRUB	23.10.1995
31	Atlantis	05.06.1997	191	MOSTALWAR	14.10.1993	351	IPSA	02.01.1987	507	EURSDG	02.01.1981
32	Atlasest	12.02.2008	192	MUZA	08.04.1998	352	JCI	04.04.1983	508	EURSEK	04.01.1971
33	ATM	28.09.2004	193	MWTRADE	28.09.2007	353	KLCI	03.01.1977	509	EURTHB	04.01.1993
34	ATMGRUPA	09.01.2004	194	NETIA	15.06.2000	354	KOSPI	01.05.1981	510	EURTRY	02.01.1991
35	ATREM	18.12.2008	195	NEUCA	30.09.2004	355	MDAX	29.02.1996	511	EURTWD	02.01.1991
36	AWBUD	01.06.1998	196	NEWAG	05.12.2013	356	MERVAL	16.05.1988	512	EURUAH	02.09.1996
37	BAHOLDING	19.06.2008	197	NORTCOAST	10.04.2006	357	MEXICIPC	20.12.1993	513	EURXAG	02.01.1991
38	BALTONA	30.06.2011	198	NOVITA	16.12.1994	358	MOEX	23.09.1997	514	EURXAU	02.01.1991
39	BEDZIN	08.12.1998	199	NOWAGALA	01.07.2004	359	NASDAQ	03.05.1971	515	EURXDR	02.01.1991
40	BENEFIT	21.04.2011	200	NTTSYSTEM	12.04.2007	360	NASDAQ 100	01.10.1985	516	EURXPD	02.01.1991
41	BEST	30.05.1997	201	ODLEWNIE	12.03.1998	361	NIFTY 50	06.11.1995	517	EURXPT	02.01.1991
42	BETACOM	02.03.2004	202	OEX	20.10.2005	362	NIKKEI	01.06.1949	518	EURZAR	02.01.1991
43	BIOMEDLUB	29.07.2011	203	OPENFIN	05.04.2011	363	NZ 50	03.01.2001		**JPYXXX**	
44	BIOTON	16.03.2005	204	OPONEO.PL	12.09.2007	364	OMX RIGA	03.01.2000	519	JPYCNY	02.01.1991
45	BOGDANKA	25.06.2009	205	OPTEAM	11.10.2010	365	OMX S30	01.10.1986	520	JPYHKD	02.01.1991
46	BORYSZEW	20.05.1996	206	ORANGEPL	18.11.1998	366	OMX STOCKHOLM	30.09.1986	521	JPYIDR	25.10.1993
47	BOS	03.02.1997	207	ORBIS	20.11.1997	367	OMX TALLINN	03.01.2000	522	JPYILS	25.10.1993
48	BOWIM	25.01.2012	208	ORCOGROUP	21.06.2007	368	OMX VILNIUS	03.01.2000	523	JPYINR	02.01.1991
49	BRASTER	20.12.2012	209	ORION	09.02.2009	369	OSE ALL	03.01.1983	524	JPYKRW	02.01.1991
50	BSCDRUK	04.01.2011	210	ORZBIALY	22.10.2007	370	PSEI	02.01.1986	525	JPYMYR	02.01.1991
51	BUDIMEX	25.05.1995	211	OTLOG	30.08.2013	371	PSI 20	03.01.1983	526	JPYNZD	02.01.1991
52	BUMECH	14.01.2009	212	OTMUCHOW	29.09.2010	372	PX	07.09.1993	527	JPYPHP	25.10.1993

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53	CCC	02.12.2004	213	OVOSTAR	29.06.2011	373	RTSI	04.09.1995	528	JPYSGD	02.01.1991
54	CDPROJEKT	02.08.1994	214	PAMAPOL	13.12.2013	374	RUSSEL	22.10.2001	529	JPYTHB	04.01.1993
55	CDRL	10.12.2014	215	PANOVA	20.07.2007	375	S&P ASX 200	01.06.1992	530	JPYTWD	02.01.1991
56	CELTIC	23.12.2010	216	PATENTUS	09.11.2009	376	S&P TSX COMPOSITE	03.01.1977	531	JPYBGN	23.10.1995
57	CEZ	25.10.2006	217	PCCEXOL	03.08.2012	377	SASESLCT	02.01.2003	532	JPYCHF	04.01.1971
58	CIECH	10.02.2005	218	PCCROKITA	25.06.2014	378	SAX	03.07.1995	533	JPYCZK	23.10.1995
59	CIGAMES	30.11.2007	219	PEKAO	30.06.1998	379	SDAX	15.03.1999	534	JPYDKK	02.01.1991
60	CNT	06.11.1998	220	PEP	13.05.2005	380	SET	05.01.1982	535	JPYGBP	04.01.1971
61	COGNOR	19.03.1997	221	PEPEES	19.05.1997	381	Shanghai Composite	20.12.1990	536	JPYHRK	23.10.1995
62	COMARCH	10.03.1999	222	PFLEIDER	06.05.1997	382	SMI	01.07.1988	537	JPYHUF	23.10.1995
63	COMP	14.01.2005	223	PGE	06.11.2009	383	SOFIX	26.11.2001	538	JPYISK	23.10.1995
64	CORMAY	20.08.2008	224	PGNIG	23.09.2005	384	SP500	17.02.1885	539	JPYNOK	02.01.1991
65	CPGROUP	13.12.2013	225	PGSSOFT	29.10.2008	385	STRAITS TIMES	28.12.1987	540	JPYPLN	16.05.1991
66	CYFRPLSAT	06.05.2008	226	PHARMENA	25.08.2008	386	TA 35	20.10.1992	541	JPYRON	23.10.1995
67	DEBICA	24.11.1994	227	PHN	13.02.2013	387	TAIEX	05.01.1995	542	JPYRUB	23.10.1995
68	DECORA	21.06.2005	228	PKNORLEN	26.11.1999	388	TECDAX	16.09.1999	543	JPYSEK	04.01.1971
69	DELKO	18.09.2009	229	PKOBP	10.11.2004	389	TOPIX	22.10.2001	544	JPYTRY	02.01.1991
70	DEVELIA	29.06.2007	230	PKPCARGO	30.10.2013	390	TSE300	15.08.1989	545	JPYUAH	02.09.1996
71	DGA	20.04.2004	231	PLASTBOX	27.02.2004	391	UK 100	13.11.1935	546	JPYBRL	03.01.1995
72	DOMDEV	24.10.2006	232	PMPG	22.12.1995	392	UX	03.11.1997	547	JPYCAD	04.01.1971
73	DREWEX	15.05.2008	233	POLICE	14.07.2005	393	WIG	16.04.1991	548	JPYCLP	25.10.1993
74	DROP	04.12.2007	234	POLIMEXMS	15.10.1997	394	XU 100	02.01.1990	549	JPYMXN	02.01.1991
75	DROZAPOL	23.12.2004	235	POLNORD	18.12.1998		**Commodities**		550	JPYEGP	16.11.1995
76	ECHO	05.03.1996	236	POZBUD	11.06.2008	395	BRENT	27.06.1988	551	JPYNAD	02.01.1991
77	EDINVEST	27.10.2010	237	PRAGMAFA	14.06.2007	396	CANOLA	03.09.1974	552	JPYZAR	02.01.1991
78	EFEKT	22.04.1993	238	PRAGMAINK	14.04.2008	397	CO_2 (ICE)	01.02.2008	553	JPYXAG	02.01.1991
79	EKOEXPORT	16.07.2009	239	PROCHEM	30.06.1994	398	COAL (ROTTERDAM)	18.08.2006	554	JPYXAU	02.01.1991
80	ELBUDOWA	09.02.1996	240	PROJPRZEM	10.08.1999	399	COCOA	27.12.1979	555	JPYXPD	02.01.1991

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81	ELEKTROTI	11.04.2007	241	PROTEKTOR	14.07.1998	400	COFFEE (ARABIKA)	01.09.1972	556	JPYXPT	02.01.1991
82	ELEMENTAL	12.07.2012	242	PROVIDENT	27.03.2013	401	COFFEE (ROBUSTA)	31.07.2008	557	JPYXDR	02.01.1991
83	ELKOP	07.03.2001	243	PULAWY	19.10.2005	402	COPPER	01.07.1959	558	JPYAUD	02.01.1984
84	ELZAB	13.05.1998	244	PZU	12.05.2010	403	CORN	02.07.1959		**Other Currency Pairs**	
85	EMCINSMED	29.07.2005	245	QUANTUM	16.08.2007	404	COTTON	01.07.1959	559	AUDCHF	02.01.1980
86	ENAP	20.08.1997	246	QUERCUS	11.09.2008	405	ETHANOL	24.03.2005	560	AUDNZD	02.01.1991
87	ENEA	17.11.2008	247	RADPOL	10.05.2007	406	FEDDER CATTLE	30.11.1971	561	AUDCAD	02.01.1980
88	ENELMED	14.06.2011	248	RAFAKO	07.03.1994	407	GAS OIL (LONDYN)	03.05.2007	562	CADCHF	04.01.1971
89	ENERGA	11.12.2013	249	RAFAMET	25.06.2007	408	GASOLINE RBOB	04.10.2005	563	GBPAUD	02.01.1980
90	ENERGOINS	25.09.2007	250	RAINBOW	09.10.2007	409	GOLD	02.06.1969	564	GBPCAD	04.01.1971
91	ERBUD	11.05.2007	251	RANKPROGR	08.07.2010	410	HEATING OIL	02.01.1980	565	GBPCHF	04.01.1971
92	ERG	18.02.2005	252	RAWLPLUG	30.11.2004	411	INDEX CRB	03.01.1994	566	GBPPLN	16.05.1991
93	ERGIS	12.06.2006	253	RELPOL	01.02.1996	412	LEAD	07.07.2008	567	NZDJPY	02.01.1991
94	ESOTIQ	16.06.2011	254	REMAK	23.11.1994	413	LEAN HOGS	28.02.1966	568	GBPNZD	09.01.1984
95	ESSYSTEM	09.03.2007	255	RONSON	05.11.2007	414	LIVE CATTLE	30.11.1964	569	AUDPLN	16.05.1991
96	EUCO	29.12.2010	256	ROPCZYCE	16.12.1997	415	LUMBER	27.12.1979	570	CHFPLN	16.05.1991
97	EUROCASH	04.02.2005	257	RUBICON	12.06.1997	416	MILK CLASS III (CME)	11.01.1996			
98	EUROTEL	29.12.2006	258	SANOK	16.01.1997	417	NATURAL GAS	04.04.1990			
99	EVEREST	26.11.2014	259	SANPL	22.06.1993	418	NICKEL	07.07.2008			
100	FAMUR	04.08.2006	260	SANWIL	12.02.1998	419	OAT	01.07.1959			
101	FASING	18.09.2000	261	SARE	25.03.2011	420	ORANGE JUICE (FROZEN)	01.02.1967			
102	FERRO	14.04.2010	262	SECOGROUP	05.12.2007	421	PALLADIUM	03.01.1977			
103	FERRUM	27.06.1997	263	SEKO	15.03.2007	422	PLATINUM	01.03.1968			
104	FORTE	24.07.1996	264	SELENAFM	18.04.2008	423	RAPESEED	17.09.2001			
105	GETIN	10.05.2001	265	SELVITA	14.07.2011	424	ROUGH RICE	02.01.1987			
106	GETINOBLE	20.01.2012	266	SESCOM	22.05.2013	425	SILVER	13.06.1963			
107	GLCOSMED	19.08.2013	267	SETANTA	08.01.2013	426	SOYBEAN MEAL	17.07.1959			

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108	GPW	09.11.2010	268	SFINKS	08.06.2006	427	SOYBEAN OIL	27.12.1979			
109	GROCLIN	24.11.1998	269	SILVANO	23.07.2007	428	SOYBEANS	01.07.1959			
110	GRODNO	13.04.2011	270	SIMPLE	14.12.2000	429	SUGAR #11	04.01.1961			
111	GRUPAAZOTY	30.06.2008	271	SKARBIEC	18.11.2014	430	TIN	07.07.2008			
112	GTC	06.05.2004	272	SKOTAN	11.01.1999	431	ULSD NY HARBOR (NYMEX)	15.11.1978			
113	HANDLOWY	30.06.1997	273	SKYLINE	04.04.2008	432	WHEAT (HARD RED)	05.01.1970			
114	HELIO	16.04.2007	274	SNIEZKA	31.12.2003	433	WHEAT (SPRING)	08.08.1972			
115	HMINWEST	07.11.2013	275	SOHODEV	12.06.1997	434	WTI	31.03.1983			
116	HOLLYWOOD	24.03.2014	276	SOLAR	19.04.2012	435	ZINK	18.02.2008			
117	HYDROTOR	17.03.1998	277	SONEL	23.06.2008		**FX market**				
118	IFFIRMA	20.03.2008	278	STALEXP	26.10.1994		**USDXXX**				
119	IMCOMPANY	04.05.2011	279	STALPROD	06.08.1997	436	USDAUD	04.01.1971			
120	IMMOBILE	11.07.2007	280	STALPROFI	26.04.2000	437	USDBGN	23.10.1995			
121	IMPEL	16.10.2003	281	SUNEX	29.09.2011	438	USDBRL	02.01.1995			
122	IMPERA	12.06.1997	282	SUWARY	14.10.1998	439	USDCAD	04.01.1971			
123	IMS	04.01.2012	283	SWISSMED	15.10.2004	440	USDCHF	04.01.1971			
124	INC	20.07.2006	284	SYGNITY	27.10.1995	441	USDCLP	25.10.1993			
125	INDATA	20.09.2010	285	SYNEKTIK	09.08.2011	442	USDCNY	09.01.1984			
126	INDYKPOL	12.10.1994	286	TALEX	20.11.2000	443	USDCZK	23.10.1995			
127	INGBSK	25.01.1994	287	TATRY	15.10.2012	444	USDDKK	02.01.1984			
128	INPRO	17.02.2011	288	TAURONPE	30.06.2010	445	USDEGP	16.11.1995			
129	INSTALKRK	28.04.1999	289	TERMOREX	01.10.2012	446	USDEURO	04.01.1971			
130	INTERAOLT	18.12.2012	290	TESGAS	08.07.2009	447	USDGBP	04.01.1971			
131	INTERCARS	26.05.2004	291	TIM	16.02.1998	448	USDHKD	09.01.1984			
132	INTERFERI	10.08.2006	292	TORPOL	05.09.2014	449	USDHRK	23.10.1995			
133	INTERSPPL	11.07.2006	293	TOYA	12.08.2011	450	USDHUF	23.10.1995			
134	INTROL	08.11.2007	294	TRAKCJA	01.04.2008	451	USDIDR	25.10.1993			

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135	INVISTA	24.01.2008	295	TRANSPOL	12.09.2008	452	USDILS	25.10.1993			
136	IPOPEMA	26.05.2009	296	TRITON	12.11.1998	453	USDINR	02.01.1973			
137	ITMTRADE	10.08.2011	297	ULMA	21.05.1997	454	USDISK	23.10.1995			
138	IZOBLOK	22.12.2011	298	UNIBEP	08.04.2008	455	USDJPY	21.01.1974			
139	IZOLACJA	23.10.2007	299	UNIMA	13.09.2006	456	USDKRW	13.04.1981			
140	IZOSTAL	11.01.2011	300	UNIMOT	26.09.2012	457	USDMXN	09.11.1989			
141	JHMDEV	09.08.2011	301	URSUS	27.12.2007	458	USDMYR	08.01.1990			
142	JSW	06.07.2011	302	VINDEXUS	23.03.2009	459	USDNAD	09.01.1984			
143	JWCONSTRUCTION	04.06.2007	303	VIVID	11.06.2012	460	USDNOK	04.01.1971			
144	K2INTERNT	24.04.2008	304	VOTUM	20.12.2010	461	USDNZD	04.01.1971			
145	KANIA	22.12.2008	305	VOXEL	11.10.2011	462	USDPHP	25.10.1993			
146	KERNEL	23.11.2007	306	VRG	30.09.1993	463	USDPLN	16.05.1991			
147	KETY	30.01.1996	307	WADEX	03.06.2008	464	USDRON	23.10.1995			
148	KGHM	10.07.1997	308	WARIMPEX	29.01.2007	465	USDRUB	23.10.1995			
149	KINOPOL	12.04.2011	309	WASKO	24.05.2001	466	USDSDG	02.01.1981			
150	KOGENERA	26.05.2000	310	WAWEL	11.03.1998	467	USDSEK	04.01.1971			
151	KOMPAP	19.11.1996	311	WIELTON	28.11.2007	468	USDTHB	04.01.1993			
152	KOMPUTRON	09.07.2007	312	WIKANA	20.02.1997	469	USDTRY	02.01.1984			
153	KONSSTALI	06.12.2007	313	WOJAS	02.04.2008	470	USDTWD	02.01.1984			
154	KPPD	17.10.2007	314	WORKSERV	26.04.2012	471	USDUAH	02.09.1996			
155	KRAKCHEMIA	23.08.2007	315	YOLO	29.07.2011	472	USDXAG	09.01.1984			
156	KREC	14.02.2008	316	ZAMET	10.10.2011	473	USDXAU	09.01.1984			
157	KREDYT INKASO	11.06.2007	317	ZEPAK	30.10.2012	474	USDXDR	02.01.1991			
158	KRKA	10.02.1997	318	ZPUE	23.08.2000	475	USDXPD	09.01.1984			
159	KRUK	10.05.2011	319	ZUE	01.10.2010	476	USDXPT	09.01.1984			
160	KRUSZWICA	06.01.1997	320	ZYWIEC	24.09.1991	477	USDZAR	04.01.1971			

Source: own study.