Maciej K. Dudek[±]

Volatility as a Choice

Abstract

Human beliefs, while always remaining in equilibrium, serve as an equilibrium selector and determine the degree of aggregate volatility. Fully *rational* and risk averse economic agents expect macro-level dynamics to be characterised by a specific degree of volatility. Given this expectation the agents respond *rationally* by building up higher buffer stock savings in response to *perceived* volatility. The economy, given the change in individual behaviour, responds, the process of physical capital formation is endogenously altered, and it displays volatility that is in line with the initial expectation of *rational* economic agents. As a result, the beliefs, while being self-confirming, determine endogenously the degree of volatility at the aggregate level.

Keywords: endogenous volatility, self-confirming beliefs, general equilibrium, rationality

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[±] Vistula University and Polish Academy of Science
Please send comments to mkdudek@alumni.caltech.edu.

1. Introduction

Economic data displays a varying degree of volatility. At times time series data produced by economic systems is characterised by a very small variance only to experience an unexpected transformation to a new *regime* characterised with a much higher variance. Examples and explanations are abundant ranging from time series data generated by financial markets and exchange rate markets, to aggregate macro-level dynamics. In this paper, we use a tractable general equilibrium model to explain why the degree of volatility of an economy can be time-varying. More importantly, we argue that the volatility is fully endogenous and it reflects private choices of *rational* economic agents. Specifically, we show that the perceptions of *rational* economic agents with regard to the volatility of the economy can serve as an equilibrium selector. In particular, we argue that if *rational* economic agents believe that the economy exhibits a significant degree of volatility, then they react, given their beliefs, appropriately and build up buffer stock savings. This endogenously effects the process of physical capital formation and in turn resultant macro-level dynamics. Naturally, we ensure that the beliefs form an equilibrium as well, ie we show that the ensuing macro-level dynamics display, in equilibrium, the degree of volatility equal to that originally expected by *rational* economic agents. On the other hand, we show that if *rational* economic agents expect the economy to be stable and the future to be predictable, then they respond *rationally* and choose not to build up buffer stock savings, which endogenously affects the process of physical capital formation and in turn the dynamics at the macro-level. We show, in this case, that the resulting macro-level dynamics is characterised by virtually zero variance verifying the original beliefs of *rational* economic agents and ensuring that the beliefs are in equilibrium. In other words, we show in this paper that the degree of volatility of an economy is endogenous and it can be chosen within the system by *rational* economic agents.

The approach presented in this paper differs substantively from that, exemplified in a modern and sophisticated treatment of Bloom (2009), employed in the rest of the literature. Specifically, we model the process of the selection of volatility endogenously and we do not rely on exogenously given probability distributions to trigger the transition from one *regime* to another. More importantly, in our model the degree of volatility is selected consciously by *rational* economic agents and is not driven by a specific choice of the underlying parameters. In other words, the observed volatility in our model reflects the private choices, in response to *perceived* volatility, of actors operating within the system and not a peculiar draw of values for the underlying parameters. Naturally, we formally construct an equilibrium and ensure that the *actual* and the *perceived* degrees of volatility coincide. Furthermore, in this paper we present a novel mechanism that allows the degree of volatility to be determined within the system without relying on the existence of exogenous coordination devices and, thus, our approach can be viewed as distinct and at the same time complementary to that described in the sunspot literature.

We cast our results in a simple framework that allows for endogenous determination of the space of beliefs of economic agents. To achieve our results, we allow without ever departing from complete *rationality* for the possibility of endogenous instability in our model. This is done by introducing *naive* agents, originally described by Grossman & Stiglitz (1980), into the *space of beliefs* held by *rational* agents. Formally, in our model all agents at all times are fully *rational*. However, the assumption of common knowledge of *rationality* is relaxed, and *rational* economic agents are allowed to *presuppose*, and in equilibrium they do, that some other agents can be *naive*. Naturally, in equilibrium all agents are *rational* and no agent is *naive*, as *naive* agents exist only in the *space of beliefs* of *rational* agents, not in reality. Naturally, in equilibrium, the absence from reality of *naive* agents cannot be detected by *rational* agents as in equilibrium the observables behave as if *naive* agents were present even though in reality they do not exist.

In the conceptual sense, our paper is closely related to a very recent and elegant contribution by Eyster & Piccione (2012) who study asset pricing when economic agents are boundedly *rational* and possess an incomplete picture of the economy, but at the same time remain statistically *correct* and their beliefs with regard to values of prices conform to those actually observed. In this paper, we utilise a similar approach; our agents form beliefs with regard to the operational structure of the economy; given the beliefs, they act *rationally* and their actions form reality. Moreover, given the beliefs, *rational* economic agents are able to derive the *perceived* and, at the same time, complete structure of the economy, ie our agents know the model. Furthermore, we show that the dynamics generated by an economy described with the *perceived* structure can be identical to that observed in reality. In that sense, the beliefs of *all* agents in our model are verified in equilibrium and remain statistically correct at all times. Furthermore, the critical distinction between our model and that of Eyster & Piccione (2012) remains as agents in their model are *boundedly* rational and agents in our model are *fully* rational.

In a similar vein, our contribution differs from that of Kurz & Motolese (2011) who explain the endogenoity of risk premia, but rely on a framework with heterogeneous beliefs and market dynamics, which is by assumption too complex to be learned by economic agents. In this paper, we show, however, that volatility can be endogenous without limiting the learning abilities of economic agents. Moreover, in our model, agents, in equilibrium, hold identical beliefs.

Recently, Eusepi & Preston (2011), who study a statistically correct feedback mechanism between private decisions at the micro-level and aggregate dynamics, contributed a sophisticated model that allows shifts in expectations of economic agents to account for macro-level dynamics and the expectations themselves to be partially validated. The model presented in this paper shares similar features; nevertheless, the key difference remains. In our model economic agents are *fully* rational and are able to derive the correct link between their private actions and aggregate dynamics, whereas in their contribution, economic agents are *boundedly* rational and do not understand the link between their private actions at the micro-level and aggregate dynamics and must exclusively rely on statistical means to verify the consistency of their beliefs.

The contribution of Calvet (2001) shows how different levels of volatility can arise endogenously in simple OLG economies under varying degrees of market incompleteness. The results of Calvet, however, hinge on the values of the underlying parameters being numerically proper. In this paper, on the other hand, we show that actors who operate within the system can influence the degree of aggregate volatility by taking specific and equilibrium-consistent actions.

Our contribution from a formal perspective provides a constructive proof that the notion of a self-fulfilling mistake originally described by Grandmont (1998) canfind support in a *rational* framework. Specifically, we show that economic agents, by expecting a given degree of volatility of aggregate dynamics, can adjust their private actions accordingly and can in fact influence aggregate activity sufficiently to ensure that the observables conform to the expected degree of volatility. In that sense, agents in our model are always *correct* as their beliefs imply and remain consistent with the observables. However, at the same time, agents in our model *err* as their beliefs do not correspond to the objective truth, as agents in our model, in equilibrium, *presuppose* that some other agents are *naive* even though that it is not the case.

The main findings of our paper can be thought of as an extension of the results obtained by Sorger (1998) who shows that the form of macro-level dynamics can be selected by expectations of economic agents. Specifically, Sorger argues that the path followed by the interest rate can be endogenously shaped by the beliefs and belief consistent actions of economic agents, and it can exhibit random behaviour if economic agents expect it to be random. In this paper we obtain analogous results, but with regard to the degree of the volatility of an economy rather than with regard to the path of the interest rate. More importantly in Sorger's contribution, agents are *boundedly rational*, whereas our results hold in a fully *rational* framework. Formally, Sorger derives his results under a weaker equilibrium concept, *CEE* as defined by Hommes (1998); our results, on the other hand, hold in a *REE*, as originally outlined by Lucas (1972).

From the technical point of view, our results are obtained in a general equilibrium model based on Matsuyama (1999) and Dudek (2010). Moreover, our contribution extends the findings of Dudek (2012) and shows that more profound results hold if one allows for risk aversion of economic agents. Specifically, we show that *perceived* volatility determines actions of risk averse and *rational* individuals. Furthermore, we show that the impact of the perceived volatility on individual actions and the ensuing equilibrium dynamics can be sufficient to ensure that the economy exhibits *actual* volatility identical to the perceived one, and, thus, ensuring that the beliefs of economic agents are in equilibrium and at the same time determine the degree of aggregate volatility.

In addition, we want to emphasize that our contribution touches on a different point than that brought in more traditional approaches to the issue of endogenous instability and time-varying volatility. Normally, authors, see Brock & Hommes (1997), make a point that the economy can be temporarily attracted to a given region and display low volatility only to escape, without any external stimuli, to a different basin of attraction where the displayed volatility is much higher. Such a process of endogenous switching can continue indefinitely, and the observed volatility can be time-varying even if shocks do not occur. In this paper, we make a drastically different point. Our results do not rest on the properties of the underlying dynamical system; they are driven by conscious actions of *rational* and fully optimising agents who form expectations, verified in equilibrium, with regard to the degree of volatility.

There are in total six sections in the paper. Section two outlines the model. The following section defines the equilibrium. Consistency of beliefs is established in section four. Additional results are discussed in section five. Finally, section six concludes.

2. Model

We cast our nding in a standard general equilibrium macro model. In particular, we rely on a version of the Diamond (1965) OLG model with a continuum of measure one of agents entering the economy each period. We assume that the preferences of agent $i \in [0,1]$ born at time t are represented with the following utility function:

$$U\left(c_{1,t}^{i}, c_{2,t+1}^{i}\right) = -e^{-c_{1t}^{i}} - \beta_{it} e^{-c_{2t+1}^{i}}, \tag{1}$$

where $\beta_{i,t}$ is a random variable independent across time and across agents, representing a preference shock of agent $i \in [0,1]$ at time *t*. To shorten notation we denote $log\left(\beta_{i,t}\right)$ with $\varepsilon_{i,t}$. Furthermore, we assume that each period $\varepsilon_{i,t}$ is independently drawn for each agent from distribution $F_{\varepsilon}(\,)$ such that:

$$\int_{-\infty}^{\infty} \varepsilon_{i,t}\, dF_{\varepsilon}\left(\varepsilon_{i,t}\right) = \bar{\varepsilon}. \tag{2}$$

In addition, we assume that a given agent born at time *t* earns income $y_{,1t}$, which can be thought of as labor income in the first period of her life, and that she receives income $y_{2,t+1}$, which can be interpreted as profit income in the second period of her life. Naturally, the agent earns, in the second period, a return on her saving. Consequently, we can express the relevant budget constraints as:

$$\begin{gathered} p_{1,t} c_{1,t}^{i} + s_{t}^{i} = y_{1,t} \\ p_{t}^{k} K_{t+1}^{i} = s_{t}^{i} \\ p_{2,t+1} c_{2,t+1}^{i} = \left(p_{t+1}^{k}(1-\delta) + r_{t+1}\right) K_{t+1}^{i} + y_{2,t+1}, \end{gathered} \tag{3}$$

where $p_{1,t}$ denotes the price of a unit of consumption, valued by young agents, in period *t*, $p_{2,t+1}$ denotes the price of a unit of consumption, valued by old agents, at time *t* + 1; r_{t+1} denotes the rental price of capital at time *t* + 1, and finally p_{t}^{k} and p_{t+1}^{k} denote the prices of a unit of physical capital at time *t* and *t* + 1, respectively. Observe that we have assumed that physical capital is the only saving instrument.

It is our desire to present our results in the simplest framework possible even though our underlying problem is fundamentally non-trivial as it involves searching for a fixed point in an environment characterised by heterogeneous beliefs. Accordingly, to preserve analytic tractability of the model and to ensure that the key equilibrium variables can be expressed with closed form solutions, we assume that physical capital depreciates fully after one period, ie that:

$$\delta = 1. \tag{4}$$

The problem of agent *i* born at time *t* is to form assessments of her future income and to choose the optimal amount saved at time *t*. In general, agents in our model must deal with uncertainty as the future incomes are not known in advance. Consequently, we assume that agents are expected utility maximisers. Now, given the assumptions we can express the problem of an agent $i \in [0,1]$ born at time *t* whose information set is denoted with:

$$max_{\{K_{t+1}^i\}} E\left[U\left(c_{1,t}^i, c_{2,t+1}^i\right)|\Omega_t^i\right] = -e^{-c_{1t}^i} - \beta_{it} E\left[e^{-c_{2t+1}^i}|\Omega_t^i\right], \quad (5)$$

s.t.

$$c_{1,t}^i = \frac{y_{1,t}}{p_{1,t}} - \frac{p_t^k}{p_{1,t}} K_{t+1}^i$$
$$c_{2,t+1}^i = \frac{r_{t+1}}{p_{2,t+1}} K_{t+1}^i + \frac{y_{2,t+1}}{p_{2,t+1}}. \quad (6)$$

Obviously, we can rewrite the problem described with equations (5) and (6) in an equivalent form as:

$$max_{\{K_{t+1}^i\}} - e^{-\left(\frac{y_{1,t}}{p_{1,t}} - \frac{p_t^k}{p_{1,t}} K_{t+1}^i\right)} - \beta_{it} E\left[e^{-\left(\frac{r_{t+1}}{p_{2,t+1}} K_{t+1}^i + \frac{y_{2,t+1}}{p_{2,t+1}}\right)}|\Omega_t^i\right]. \quad (7)$$

Naturally, the relevant first order condition is given by:

$$\frac{p_t^k}{p_{1,t}} e^{-\left(\frac{y_{1,t}}{p_{1,t}} - \frac{p_t^k}{p_{1,t}} K_{t+1}^i\right)} = \beta_{it} E\left[\frac{r_{t+1}}{p_{2,t+1}} e^{-\left(\frac{r_{t+1}}{p_{2,t+1}} K_{t+1}^i + \frac{y_{2,t+1}}{p_{2,t+1}}\right)}|\Omega_t^i\right]. \quad (8)$$

The above efficiency condition defines implicitly the optimal amount saved by agent *i* at time *t*. The condition appears be, in general, non-tractable. However, the specification of the supply side in our model, based on Matsuyama (1999) (see Dudek (2010) for details), allows us to derive a closed form solution for K_{t+1}^i.

Observe that to solve for K_{t+1}^i, agent *i* must form assessments of the values of two period *t* + 1 variables. In particular, the agent at time *t* must assess the real value of the future real rental price of capital, $\frac{r_{t+1}}{p_{2,t+1}}$, and the real value of her future income, $\frac{y_{2,t+1}}{p_{2,t+1}}$. Typically, those future variables of interest are complicated functions of future fundamentals. However, in our model, again based on Matsuyama (1999) and described in detail by Dudek (2010), the relevant expressions take a very simple form. Specifically, we have:

$$\forall t: \frac{r_{t+1}}{p_{2,t+1}} = \Lambda \tag{9}$$

and

$$\forall t: \frac{y_{2,t+1}}{p_{2,t+1}} = B\Lambda K_{t+1}, \tag{10}$$

where Λ and B are constants and K_{t+1} denotes the value of period t + 1 capital stock, given in equilibrium by $K_{t+1} = \int_0^1 K_{t+1}^j \, dj$. Naturally, the functional form of (10) reveals that the aggregate production function assumes a linear form at a certain stage of production, again see Matsuyama (1999) and Dudek (2010) for a detailed description.

Furthermore, the specification of the supply side in our model allows us to establish that:

$$\forall t: \frac{p_t^k}{p_{1,t}} = 1 \tag{11}$$

and

$$\forall t: \frac{y_{1,t}}{p_{1,t}} = AK_t^{\alpha}, \tag{12}$$

where A is a constant and $\alpha \in (0,1)$. Naturally, the functional form of (12) reveals that the aggregate production function assumes a Cobb-Douglas form at a certain stage of production, again see Matsuyama (1999) and Dudek (2010) for a detailed description.

Obviously, our assumption of complete *rationality* implies, in particular, that all agents at all times are aware that properties (9), (10), (11), and (12) hold, which can be formally stated as:

$$\forall i, t: \left\{ \frac{r_{t+1}}{p_{2,t+1}} = \Lambda, \frac{y_{2,t+1}}{p_{2,t+1}} = B\Lambda K_{t+1}, \frac{p_t^k}{p_{1,t}} = 1, \frac{y_{1,t}}{p_{1,t}} = AK_t^{\alpha} \right\} \subset \Omega_t^i, \tag{13}$$

which in turn allows us to express the first order condition, (8), as:

$$e^{-(AK_t^{\alpha} - K_{t+1}^i)} = \beta_{it} E\left[\Lambda e^{-(\Lambda K_{t+1}^i + B\Lambda K_{t+1})} | \Omega_t^i\right]. \tag{14}$$

Now, using simple properties of the exponential function, we can, noting that $K_{t+1}^i \in \Omega_t^i$, rearrange equation (14) to:

$$e^{-AK_t^{\alpha}} e^{K_{t+1}^i} = \beta_{it} \Lambda e^{-\Lambda K_{t+1}^i} E\left[e^{-B\Lambda K_{t+1}} | \Omega_t^i\right], \tag{15}$$

which now allows us to identify K_{t+1}^i as, recall that $\varepsilon_{i,t} = log(\beta_{i,t})$,

$$K_{t+1}^i = \frac{1}{1+\Lambda}\{\varepsilon_{i,t} + log(\Lambda) + AK_t^\alpha + log(E[e^{-B\Lambda K_{t+1}}|\Omega_t^i])\}. \qquad (16)$$

In what follows we assume for purely aesthetic reasons that the value[1] of Λ, defined with equation (9) and reflecting deeper parameters of the model described in detail in Dudek (2010), is equal to 1. Consequently, we can rewrite equation (16) in a more transparent form:

$$K_{t+1}^i = \frac{1}{2}\{\varepsilon_{i,t} + AK_t^\alpha + log(E[e^{-BK_{t+1}}|\Omega_t^i])\}. \qquad (17)$$

Observe that, given the simplicity of the model, to solve explicitly for the optimal amount saved, K_{t+1}^i, agent *i* must, at time *t*, given her information set Ω_t^i, form an assessment of a single *future* variable, K_{t+1}.

In fact, $K_{t+1} = \int_0^1 K_{t+1}^j\, dj$ is determined at time *t* and, naturally, it reflects *private* saving decisions of all agents $j \in [0,1]$ taken at time *t*. Nevertheless, in this paper we assume that K_{t+1} is not known to agent $i \in [0,1]$ at time *t*; ie we have:

$$\forall t, i\colon K_{t+1} \notin \Omega_t^i. \qquad (18)$$

In other words, we assume that agents at time *t* do not know K_{t+1}; however, they are of course aware that the *actual* value of $K_{t+1} = \int_0^1 K_{t+1}^j\, dj$ reflects their *private* actions $\{K_{t+1}^j\}_{j\in[0,1]}$ taken at time *t*, and naturally they explore the link between their *private* actions and the value of the aggregate capital stock in their decision making process. Alternatively, we can state that economic agents in our model understand that *private* saving decisions at time *t* determine the aggregate capital stock at time *t* + 1. However, they do not not observe[2] *private* saving decisions of other agents at time *t* and consequently are not able to find $K_{t+1}$by a simple aggregation process, K_{t+1}

[1] Our choice of a numerical value for Λ just eliminates a constant from the equilibrium equations without affecting the main findings.

[2] It is impossible in this model to infer the amount invested by observing the relative price of capital as by assuming economic agents in our model save the unconsumed part of their purchases, and consequently the relative price is always equal to 1.

$= \int_0^1 K_{t+1}^j \, dj$, in real time at time *t*. Naturally, we assume that the value of K_{t+1} becomes common knowledge at time *t* + 1, ie we have $\forall t, i: K_{t+1} \in \Omega_{t+1}^i$.

The optimal amount saved by agent *i* is dictated by equation (16). Moreover, being fully *rational*, agent *i* is aware that analogous equations dictate the behavior of other agents $j \in [0,1]$. Consequently, agent *i* can easily derive the relationship that describes the evolution of the aggregate capital stock across all agents. In particular, we have:

$$K_{t+1} = \int_0^1 K_{t+1}^j \, dj = \frac{1}{2}\left\{\int_0^1 \varepsilon_{j,t} dj + AK_t^\alpha + \int_0^1 log\left(E\left[e^{-BK_{t+1}} | \Omega_t^i\right]\right) dj\right\}. \quad (19)$$

We have already assumed that K_{t+1} is not observable by agent *i* at time *t*. However, agent *i* is aware that relationship (19) holds. Consequently, agent *i* can attempt to infer the actual value of K_{t+1} by searching for a fixed point defined with equation (19). Naturally, finding the actual value of K_{t+1} need not be simple, as the information sets are not identical across agents, and typically requires agents to form expectations of others' expectations then averaging them out and consequently solving an algebraic equation. Such problems are not trivial as originally pointed out by Townsend (1983) and more recently by Hellwig & Veldkamp (2009). However, in this paper we assume that agents do embark on problems of that complexity.

3. Equilibrium

The evolution of the state variable is described with equation (19), and the individual behavior of agent $i \in [0,1]$ is dictated by her best response represented with equation (16). In this section, we use the two equations to find the evolution of $\{K_t\}$ in equilibrium.

Naturally, there is a well established procedure that allows to solve equation (19). Specifically, typically it is assumed that all agents are *rational* and that it is common knowledge that they are *rational*. Given those assumptions it is possible to identify, which we do below, the equilibrium path. Occasionally, authors[3] depart from the assumption of *complete rationality* and postulate dierent forms of *boundedly rational* behaviour and then solve for the equilibrium accordingly. In this paper, we take a middle ground. We always

[3] There exist numerous contributions that resort to *bounded rationality*. See Eyster & Piccione (2012), Eusepi & Preston (2011), Kurz & Motolese (2011), and Sorger (1998) for examples of approaches relevant to the topic of this paper.

adhere to the assumption of *complete rationality* on the part of all agents. However, we relax the assumption of *common knowledge* of *complete rationality* and solve for the equilibrium accordingly.

Observe that there is no extrinsic aggregate uncertainty in the model. If anything, we only have idiosyncratic noise that washes out in equilibrium subject to the qualication of Judd (1995). Therefore, it might be reasonable to assume that K_{t+1} is not random. Formally, if $\{K_{t+1}\}$ is not random and this fact is commonly known (more precisely it constitutes common knowledge) then we can simplify equation (19) to:

$$K_{t+1} = \left\{\int_0^1 \varepsilon_{j,t} dj + AK_t^{\alpha} + \int_0^1 log(e^{-BK_{t+1}})\, dj\right\}, \tag{20}$$

which in turn reduces to:

$$K_{t+1} = \frac{1}{2+B}\left\{\int_0^1 \varepsilon_{j,t} dj + AK_t^{\alpha}\right\}, \tag{21}$$

and further, noting (2), to:

$$K_{t+1} = \frac{1}{2+B}\{\bar{\varepsilon} + AK_t^{\alpha}\}. \tag{22}$$

Naturally, equation (22) confirms that indeed K_{t+1} is nonrandom, verifying the initial belief commonly held by economic agents. More importantly, equation (22) implies that the economy converges to a steady state with a fixed value of physical capital implying zero asymptotic volatility.

In other words, the solution to our problem, the form of the equilibrium, can take a particularly simple form. If economic agents are *rational*, and it is common knowledge that they perceive K_{t+1} as nonrandom from the perspective of period *t*, then indeed K_{t+1} is deterministic and converges to a steady state. However, we argue below that it is not the only possibility. In particular, we show that $\{K_t\}$ can exhibit permanent volatility consistent with agents' beliefs.

Equation (19) that determines the evolution of the value of the economy wide capital stock is affected by the beliefs of economic agents, $\{\Omega_t^j\}_{j\in[0,1]}$. Consequently, our assumptions about the structure of the beliefs are crucial for the determination of the *actual* law of motion. We have already argued that in the case when all agents are *rational* and it is common knowledge that all agents are *rational* and perceive $\{K_{t+1}\}$ as nonrandom, then equation (19) can be easily solved implying, in particular, a nonrandom value of $\{K_{t+1}\}$ validating the initial belief. However, agents while remaining fully *rational* can hold other beliefs with regard to $\{K_{t+1}\}$ and the behaviour of other agents. Specifically, a given *rational* agent, $i \in [0,1]$, who is naturally aware of the form of equation (19), can express doubts about the ability of other agents to *privately* solve equation (19), which involves heterogeneous information sets. Observe that agents in our model do not observe *private* actions of other agents. Therefore, it need not be always proper to outright believe for *rational* agent, $i \in [0,1]$, that the remaining agents are *rational* as well. If anything, *rational* agent, $i \in [0,1]$, can eventually learn that remaining agents are *rational* as well by performing proper tests on observables, series $\{K_\tau\}_{\tau \le t}$. Furthermore, if doubts with regard to *rationality* of other agents are actually expressed, then a given *rational* agent, $i \in [0,1]$, must[4] take into account her beliefs and solve accordingly for the equilibrium. Naturally, it must still be the case – given the assumed *rationality* of agent $i \in [0,1]$, that the beliefs remain consistent with the observables.

In other words, the solution to equation (19) is very simple if we assume that all agents are *rational* and it is common knowledge that they are. However, as we argue below, the solution can be much more complex in the case when one of the two assumptions is relaxed. Both assumptions: complete *rationality* on the part of all agents and the fact that it is common knowledge that all agents are *rational* have been criticized. In particular, in a recent contribution Strzałecki [22] explores how outcomes are affected if the assumption of common knowledge of *rationality* is relaxed. In this paper we follow a similar path. We adhere to the assumption of *rationality* on the part of all agents, but we choose to lift the assumption that *rationality* constitutes common knowledge[5]. Formally, we make the following assumption.

Assumption #1. At all times all agents are fully rational. *However, it is not* common knowledge *that they are.*

Let us now proceed by describing the mind set, the belief structure, of *rational* agent $i \in [0,1]$, who maintains at time *t* that:

- agents $j \in [0, \bar{x}] \cup \{i\}$ are fully *rational* as well and share her view of the world;
- agents $j \in [\bar{x}, 1] \backslash \{i\}$ are *naive* and use simplied rules to assess the values of future variables;

[4] See Caremer et al. [5] for an illustration of what occurs if that is not the case.

[5] These words are being written at the time of the 2013 North Korean crisis. Contrary to the standard practice the US policy makers are hesitant to assume that the other party is *rational*. Consequently, the *rationality* of the *other* side or its lack has become the key determinant of the American best response.

- agents $j \in [\bar{x}, 1]\backslash\{i\}$ are *naive* and are aected by a common preference shock.

Furthermore, we assume that *rational* agent $i \in [0,1]$ considers $\bar{x}$ to be a time-invariant constant common to all agents. Moreover, *rational agent* $i \in [0,1]$ believes that *naive* agents who exist in proportion $1 - \bar{x}$ find equation (19) to be too complex and are not able to solve the equation in a *rational* manner. Consequently, they resort to basic econometric exercises to assess the value of K_{t+1}. Specifically, agent *i* believes that *naive* agents each period estimate with a simple OLS technique the coefficients of the following relationship:

$$K_{\tau+1} = \overline{K} + \rho(K_\tau - \overline{K}) + v_{\tau+1}, \tag{23}$$

where $v_{\tau+1}$, for $\tau < t - 1$, denotes the error term.

Let $\widehat{\overline{K}}$ and $\hat{\rho}$ denote the estimates of $\overline{K}$ and ρ using the sample of available data. The estimates simply correspond to the sample mean and first order autocorrelation of $K_\tau - \overline{K}_{\tau<t-1}$ and are in equilibrium time-invariant. In addition, let $F_v(\,)$ be the distribution of the error term. Naturally, we assume that *rational* agent, $i \in [0,1]$, believes that *naive* agents are *good* econometricians, ie that they use a *well* specified model with the error term, $v_{\tau+1}$, uncorrelated across time.

Naturally, a given *rational* agent, $i \in [0,1]$, has the capacity needed to mentally redo the exercise of *naive* agents who are just *presupposed* to exist. Specifically, it is still true that the intertemporal choice of a given *naive* agent, $j \in [\bar{x}, 1]$, are dictated by an analog of equation (16), given below:

$$K_{t+1}^j = \tfrac{1}{2}\{\varepsilon_{j,t} + AK_t^\alpha + log\big(E\big[e^{-BK_{t+1}}|\Omega_t^j\big]\big)\}. \tag{24}$$

However, now *rational* agent, $i \in [0,1]$, believes that *naive* agents, $j \in [\bar{x}, 1]$, use the results obtained with their econometric exercises to assess the value of K_{t+1}. Consequently, we can write:

$$K_{t+1} = \widehat{\overline{K}} + \hat{\rho}\Big(K_t - \widehat{\overline{K}}\Big) + v_{\tau+1}, \tag{25}$$

which when combined with equation (24) yields:

$$K_{t+1}^j = \tfrac{1}{2}\Big\{\varepsilon_{j,t} + AK_t^\alpha + log\left(E\left[e^{-B(\widehat{\overline{K}}+\hat{\rho}(K_t-\widehat{\overline{K}})+v_{\tau+1})}|\Omega_t^j\right]\right)\Big\}. \tag{26}$$

Now, using basic properties of the exponential function we can, noting that $\widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right)$ is not random, rewrite the above relationship as:

$$K_{t+1}^{j} = \frac{1}{2}\left\{\varepsilon_{j,t} + AK_t^{\alpha} - B\left(\widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right)\right) + log\left(E\left[e^{-Bv_{\tau+1}}|\Omega_t^j\right]\right)\right\}, \tag{27}$$

which further reduces – note that we assume that all *naive* agents follow the same steps hence they share the knowledge of the distribution of $v_{\tau+1}$ – to the following:

$$K_{t+1}^{j} = \frac{1}{2}\left\{\varepsilon_{j,t} + AK_t^{\alpha} - B\left(\widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right)\right) + \sigma_{l,v}^2\right\}, \tag{28}$$

where

$$\sigma_{l,v}^2 = log\left(\int_{-\infty}^{\infty} e^{-Bv_{t+1}} dF_v\left(v_{t+1}\right)\right) \tag{29}$$

is a constant[6].

Equation (28) defines the optimal amount saved by a given *naive* agent, $j \in [\bar{x}, 1] \backslash \{i\}$. We want to emphasise again that no agent in the model actually behaves in line with equation (28), as *naive* agents whose behaviour equation (28) captures are just *presupposed* to exist. In fact, equation (28) represents the imputed, by *rational* agents, $i \in [0,1]$, behaviour of *naive* agents $j \in [\bar{x}, 1] \backslash \{i\}$ who do not exist.

Now, *rational* agent, $i \in [0,1]$, can use equation (28) to find the amount invested by all *naive* agents. Specifically, we have

$$K_{t+1}^{N} = \int_{\bar{x}}^{1} K_{t+1}^{j}\, dj = \frac{1}{2}\int_{\bar{x}}^{1} \varepsilon_{j,t} dj + \frac{1-\bar{x}}{2}\left\{AK_t^{\alpha} - B\left(\widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right)\right) + \sigma_{l,v}^2\right\}. \tag{30}$$

Recall that the objective truth by assumption, equation (2), is that $\left\{\varepsilon_{i,t} = log\left(\beta_{i,t}\right)\right\}_{i \in [0,1]}$ are *i.i.d* random disturbances such that the mean of $\varepsilon_{i,t}$

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[6] Note that typically the distribution of *v* is not normal. Hence, we cannot simplify equation (29) further. Moreover, $\sigma_{l,v}^2$ is not quite proportional to the variance of *v*, thus, the subscript *lv* rather than *v*. Nevertheless, $\sigma_{l,v}^2$ is always a constant.

is equal to $\bar{\varepsilon}$. Therefore, we can always write, subject to the qualication of Judd (1995), that:

$$\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varepsilon_{j,t}dj = \bar{\varepsilon}, \tag{31}$$

and, in turn, simplify equation (30) accordingly. In this paper, however, we follow a dierent approach. Specifically, we assume that a given *rational* agent, $i \in [0,1]$, believes that *naive* agents are impressionable and they are subject to waves of optimism and pessimism similar in nature to sentiments described by Angelatos & La'O (2012). Consequently, *rational* agents believe that the preference parameters $\{\varepsilon_{j,t}\}$ of *naive* agents rather than being *i.i.d.* in nature evolve in a correlated manner. Note that those assumptions are made on the beliefs of the *rational* agents with regard to *naive* agents who in fact do not exist. For analytical convenience we assume that *rational* agents believe that *naive* agents at a given point in time are all hit with the same preference shock. Formally, we have:

$$\forall j \in [\bar{x},1]\backslash\{i\}|\varepsilon_{j,t} = \varepsilon_t, \tag{32}$$

where ε_t is drawn from a specic distribution[7], $\Psi_\varepsilon(\,)$. Naturally, we assume that the preference shock purportedly affecting *naive* agents at time *t* is not observable by *rational* agents at time *t*, ie:

$$\forall t,i|\varepsilon_t \notin \Omega_t^i. \tag{33}$$

Furthermore, we assume that *rational* agents believe that ε_t is uncorrelated across time.

Naturally, we can now, given the *incorrect* belief of *rational* agents, translate equation (31) to:

$$\frac{1}{1-\bar{x}}\int_{\bar{x}}^{1}\varepsilon_{j,t}dj = \bar{\varepsilon}, \tag{34}$$

which allows us to express the amount invested by *naive* agents as:

---

[7] Observe that formally there is no reason to assume that $\Psi_\varepsilon(\,)$ is in anyway related to $F_\varepsilon(\,)$ as the former exists only in the space of beliefs of *rational* economic agents and the individual preference shocks of other agents are not observable.

$$K_{t+1}^{N}=\frac{1-\bar{x}}{2}\left\{\varepsilon_t+\sigma_{l,v}^2+AK_t^{\alpha}-B\left(\widehat{\bar{K}}+\hat{\rho}\left(K_t-\widehat{\bar{K}}\right)\right)\right\}. \tag{35}$$

So far we have characterised the behaviour of *naive* agents as perceived by *rational* agents. Now, we determine the actual behaviour of *rational* agents. Note that the individual investment of *rational* agent, $i\in[0,1]$, is always given by:

$$K_{t+1}^{i}=\frac{1}{2}\left\{\varepsilon_{i,t}+AK_t^{\alpha}+log\left(E\left[e^{-BK_{t+1}}|\Omega_t^i\right]\right)\right\}. \tag{36}$$

Furthermore, *rational* agent, $i\in[0,1]$, recognises, given her beliefs, that the value of the period *t* + 1 capital stock, K_{t+1}, is jointly determined by actions of *rational* and *naive* agents. Consequently, we have $K_{t+1}=K_{t+1}^R+K_{t+1}^N$, where K_{t+1}^R denotes the level of investment of *rational* agents and K_{t+1}^N denotes the level of investment of *naive* agents. Now, using equation (35) and recognising that all *rational* agents share the same view of the world, it is straightforward to establish that (see Appendix A for details) the total amount invested by rational agents – as perceived by *rational* agent, $i\in[0,1]$ – is:

$$K_{t+1}^{R}=\frac{\bar{x}}{2+B\bar{x}}\left\{\sigma_{l,\varepsilon}^2+\bar{\varepsilon}-B\frac{1-\bar{x}}{2}\sigma_{l,v}^2+\left(1-\frac{1-\bar{x}}{2}B\right)AK_t^{\alpha}+\frac{1-\bar{x}}{2}B^2\left(\widehat{\bar{K}}+\hat{\rho}\left(K_t-\widehat{\bar{K}}\right)\right)\right\}, \tag{37}$$

where

$$\sigma_{l,\varepsilon}^2=\int_{-\infty}^{\infty}e^{-B\frac{1-\bar{x}}{2}\varepsilon_t}d\,\Psi_{\varepsilon}(\varepsilon_t). \tag{38}$$

Consequently, the total investment at time *t*, according to a *rational* agent, $i\in[0,1]$, is given by $K_{t+1}=K_{t+1}^R+K_{t+1}^N$, which leads to:

$$K_{t+1}=\frac{\bar{x}}{2+B\bar{x}}\left(\sigma_{l,\varepsilon}^2+\bar{\varepsilon}\right)+\frac{1-\bar{x}}{2+B\bar{x}}\sigma_{l,v}^2+\frac{1}{2+B\bar{x}}AK_t^{\alpha}-\frac{1-\bar{x}}{2+B\bar{x}}B\left(\widehat{\bar{K}}+\hat{\rho}\left(K_t-\widehat{\bar{K}}\right)\right)+\frac{1-\bar{x}}{2}\varepsilon_t. \tag{39}$$

Naturally, now, we can write the amount invested by a single *rational* agent, $i\in[0,1]$, as:

$$K_{t+1}^{i} = \frac{1}{2}\left\{\varepsilon_{i,t} + \frac{2\sigma_{l,\varepsilon}^{2} - B\bar{x}\bar{\varepsilon} - B(1-\bar{x})\sigma_{l,v}^{2}}{2+B\bar{x}} + \frac{2-(1-\bar{x})B}{2+B\bar{x}} AK_{t}^{\alpha} + \frac{B^{2}(1-\bar{x})}{2+B\bar{x}}\left(\hat{\bar{K}} + \hat{\rho}\left(K_{t} - \hat{\bar{K}}\right)\right)\right\}. \tag{40}$$

Recall that a given *rational* agent believes that *naive* agents exist in proportion $1-\bar{x}$ and are affected by waves of optimism and pessimism modeled as a common preference shock ε_t. Accordingly, a *rational* agent, $i \in [0,1]$, believes that actions of *naive* agents destabilise the economy. Specifically, given the mindset of *rational* agent, $i \in [0,1]$, this instability manifests itself with the error term, $\frac{1-\bar{x}}{2}\varepsilon_t$, in equation (39), which describes the aggregate law of motion of state variable $\{K_t\}$. In other words, *rational* agents believe that correlated preference shocks affect the behaviour of *naive* agents and in turn feed into the aggregate dynamics. Naturally, *rational* agents, given their mindsets, take this uncertainty into account in their decision-making process. Believing that the aggregate capital stock follows a random process, *rational* economic agents build buffer stock savings. This *rational* response to perceived uncertainty is captured, in particular, by $\sigma_{l,\varepsilon}^2$ in equation (40) describing the optimal behaviour of *rational* agent, $i \in [0,1]$.

Equation (39), describing the evolution of the state variable, reflects the mindset of *rational* agent, $i \in [0,1]$. In other words, it constitutes the perceived law of motion of the aggregate variable. Nevertheless, the actual law of motion is different. Recall that by assumption all agents are *rational* and some agents are just *presupposed* to be *naive*. Therefore, the true value of the aggregate variable at time *t* + 1 is given by $K_{t+1} = \int_0^1 K_{t+1}^i di$, where K_{t+1}^i, given by equation (40), represents the investment of a single *rational* agent at time *t*. Aggregating across all agents and noting that person-specific preference shocks $\{\varepsilon_{i,t}\}$ are *i.i.d.* in nature, we can write the expression describing the actual law of motion as:

$$K_{t+1} = \frac{1}{2+B\bar{x}}\left\{\sigma_{l,\varepsilon}^{2} + \bar{\varepsilon} - B\frac{1-\bar{x}}{2}\sigma_{l,v}^{2} + \left(1 - \frac{1-\bar{x}}{2}B\right)AK_{t}^{\alpha} + \frac{1-\bar{x}}{2}B^{2}\left(\hat{\bar{K}} + \hat{\rho}\left(K_{t} - \hat{\bar{K}}\right)\right)\right\}. \tag{41}$$

The actual law of motion, equation (41), is affected by a series of parameters. Those parameters can be split into several categories. Some parameters: *B*, *A*, $\bar{\varepsilon}$, and α reflect the fundamentals of the economy, preferences, resources and technology, and are always fixed and cannot be changed by economic agents. The parameter $\bar{x}$ is different in nature. It reflects the beliefs of *rational* economic agents. Consequently, its value can be and is chosen by economic agents. Naturally, the choice of $\bar{x}$ must be such so that the beliefs remain in equilibrium themselves. In other words, $\bar{x}$ is not really a parameter but rather an equilibrium variable whose value ensures that *rational* economic agents do not want to revise their beliefs. Furthermore, $\sigma_{l,v}^2$, $\hat{\hat{K}}$ and $\hat{\rho}$ are not really parameters but rather endogenous variables whose values are determined in equilibrium and are time-invariant. Recall that $\hat{\hat{K}}$ and $\hat{\rho}$ are obtained as OLS estimates of the cofficients equation (23) using sample data, and $\sigma_{l,v}^2$ captures the degree of variation of the errors from the same equation. In other words, parameters $\sigma_{l,v}^2$, $\hat{\hat{K}}$ and $\hat{\rho}$ depend on the sample data obtained with equation (41) but at the same time influence equation (41). Consequently, they must be considered to be endogenous variables whose values are not directly affected by consumers. Finally, the parameter $\sigma_{l,\varepsilon}^2$ reflects the beliefs of *rational* agents and, consequently, *a priori* it can assume any value. However, it also affects the actual law of motion and its value must be such to ensure that values generated by the actual law of motion are consistent with the assumed value of $\sigma_{l,\varepsilon}^2$. In other words, it must be again the case that the beliefs of *rational* agents remain in equilibrium.

In particular, note that agents' perceptions with regard to the volatility of the system affect the actual law of motion as terms $\sigma_{l,\varepsilon}^2$ and $\sigma_{l,v}^2$ capture the *rational* response, which is buffer stock saving, of agents to perceived uncertainty.

Naturally, the evolution of the state variable is captured with equation (41). However, to truly complete the description of the equilibrium, we must show that the beliefs of agents remain in equilibrium as well. We embark on this task next.

4. Consistency of beliefs

Observe that, in fact, the true values of the state variable are always dictated with equation (41), which is different from the perceived law of motion given by (39). Moreover, *rational* economic agents are assumed to use standard mathematical tools to derive equation (39) and are convinced that the state variable follows (39), whereas in fact it obeys (41). Naturally, *rational* economic agents in our model are incorrect[8]. Therefore, to ensure that the economy remains in equilibrium, we must show that *rational* economic agents do not have any incentive to revise their beliefs, ie we must show that beliefs held by *rational* economic agents remain in equilibrium.

First observe that the *actual* values of the state variable follow equation (41) and are observable. Moreover, note that *rational* economic agents are not aware that equation (40) exists. Nevertheless, rational agents can, at time *t*, collect data on observables, sequence $\{\widehat{K}_\tau\}_{\tau\le t}$ and can attempt to reconcile the observables with the perceived law of motion. Specifically, at any point in time a *rational* economic agent, $i \in [0,1]$, can confront her privately derived description of reality, equation (39), with observables $\{\widehat{K}_\tau\}_{\tau\le t}$. Such a confrontation can always be successful. It suffices to assume that the sentiment shocks affecting *naive* agents, ε_t, assume proper values given by:

$$\varepsilon_t = \frac{2}{1-\bar{x}}\Big\{\widehat{K}_{t+1} - \frac{1}{2+B\bar{x}}\left[\bar{x}\left(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}\right)\right] + (1-\bar{x})\sigma_{l,v}^2 + A\widehat{K}_\tau^\alpha + (1-\bar{x})B\left(\widehat{\bar{K}} + \hat{\rho}\left(K_\tau - \widehat{\bar{K}}\right)\right)\Big\}. \tag{42}$$

Naturally, equation (42), given observables $\{\widehat{K}_\tau\}_{\tau\le t}$, implies values of ε_t and we must ensure that these values form a histogram consistent with the assumed distribution of shocks, $\Psi_\varepsilon(\,)$. Furthermore, given the assumed beliefs of *rational* agents we must show that the implied values of shocks are uncorrelated across time. Finally, we must show that the actual volatility of the system corresponds to that expected by economic agents. Specifically, recall that parameters $\sigma_{l,\varepsilon}^2$ and $\sigma_{l,v}^2$ capture the response of agents to perceived volatility and at the same time affect the actual law of motion. We must show that the true volatility of the system implies that the values of $\sigma_{l,\varepsilon}^2$ and $\sigma_{l,v}^2$ implied by the actual law of motion correspond to those assumed.

To prove that we are indeed in equilibrium we must show that the set of observables and the beliefs held by economic agents form a fixed point in a multidimensional and multilayer space. We start by describing the set of beliefs of *rational* agents and then argue that the beliefs are in equilibrium, ie that *rational* agents do not have any incentive to revise their beliefs or to alter their perceptions of the world.

Recall that *rational* agents, in particular, believe that:

[8] Furthermore, undetectable errors made by *rational* agents in our model are non-trivial as a given *rational* agent would change her behaviour and thus increase her payoff if the objective truth was revealed to her.

- at any point in time the fraction of *naive* agents is equal to $1 - \bar{x}$;
- *naive* agents are affected by a common and time-independent preference shock, ε_t, drawn from distribution $\Psi_\varepsilon(\,)$;
- *naive* agents use a simple OLS technique, equation (23), for their assessments of the relevant future variables, implying the distribution of the error term, $F_v(\,)$.

Furthermore, recall that none of the above beliefs is correct, but as we argue each of the above beliefs finds support in the data.

Let us start by assuming that the values of the fundamental parameters are given by $A = 50$, $B = 29.9285967$, $\bar{\varepsilon} = -52.0910063$ and $\alpha = \frac{1}{3}$. Now, imagine that *rational* economic agents believe that the fraction of *rational* agents in the population $\bar{x}$ is given by:

$$\bar{x} = 0.4887298. \tag{43}$$

ie that there are about 49% of *rational* agents in the population.

Moreover, imagine that rational economic agents believe that the econometric estimates obtained on sample $\{\hat{K}_\tau\}$ by *naive* agents are given by[9]:

$$\hat{\rho} = 0.3230514 \tag{44}$$

and

$$\hat{\bar{K}} = 2.609672. \tag{45}$$

Furthermore, let us assume that *rational* agents believe that the distribution of the error term, v_{t+1}, in specification (23) is such, so that the value of $\sigma_{l,v}^2$, which is determined with equation (29), is given by:

$$\sigma_{l,v}^2 = 61.8662799. \tag{46}$$

In addition, let us assume that *rational* agents believe that the distribution of the shocks that affect the preferences of *naive* agents, $\Psi_\varepsilon(\,)$, is such so that, given equation (38), the implied value of $\sigma_{l,\varepsilon}^2$ is given by:

[9] The values presented below can appear to be very special. However, the given selection of values is chosen only for illustrative purposes. In particular, the listed values lead to the steady state value of capital K^* equal to 1. Moreover, our results are robust, discussion in the following section and Appendices B and C and do not depend on peculiar values of the underlying parameters as they hold on a non-zero measure set of values.

$$\sigma_{l,\varepsilon}^2 = 396.0977657. \tag{47}$$

Observe that at this stage the values of parameters given with equations (44), (45), (46) and (47) simply reflect the beliefs of *rational* agents (all agents). Furthermore, those values, together with the values of the fundamental parameters A, B, $\bar{\varepsilon}$ and α determine the *actual* law of motion, equation (41), which generates the values of the observables $\{\widehat{K}_\tau\}$. In other words, the beliefs of agents define the *actual* law of motion, ie they determine reality. Specifically, given the beliefs, the *actual* dynamics take the form depicted in Figure 1.

The process that describes the *actual* dynamics is affected by the beliefs – captured, in particular, with equations (44), (45), (46) and (47) – of *rational* agents, and, naturally, it generates the observables, time series data, $\{\widehat{K}_\tau\}$. In turn, the observables can be used to estimate the *actual*, data driven, values of $\hat{\rho}$, $\hat{\hat{K}}$ and $\sigma_{l,v}^2$. Specifically, the estimates obtained with the *actual* data, $\{\widehat{K}_\tau\}$, are given by:

$$\hat{\hat{\rho}} = 0.3257326, \tag{48}$$

$$\hat{\hat{\hat{K}}} = 2.613967 \tag{49}$$

and

$$\hat{\sigma}_{l,v}^2 = 61.9074768. \tag{50}$$

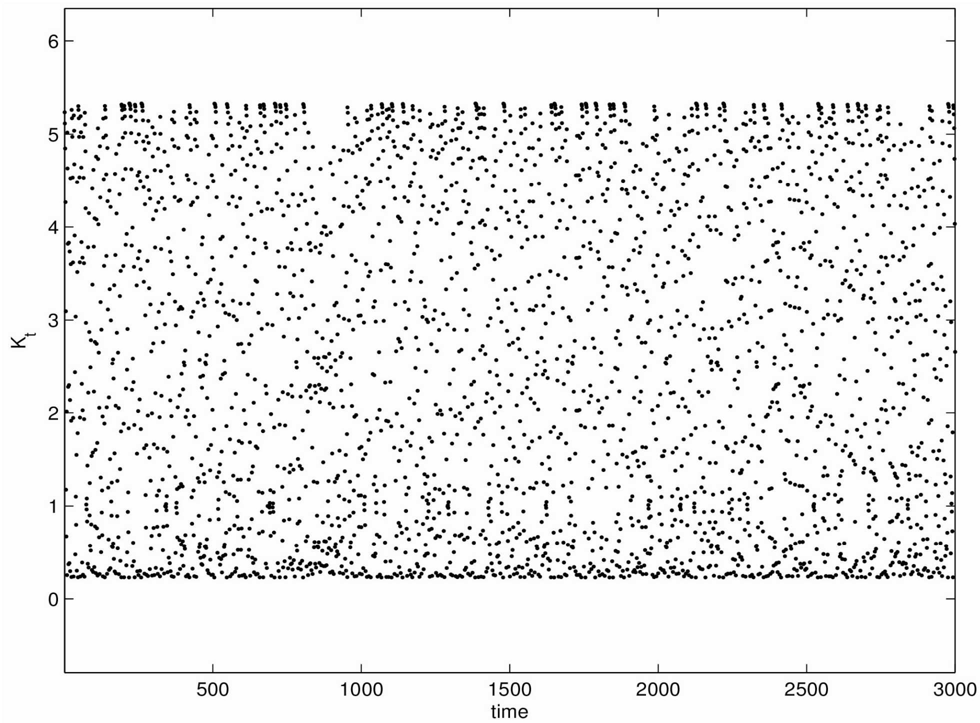

Figure 1. The Evolution of Capital Stock for T = 1000000 iterations and for $A = 50$, $B = 29.9285967$, $\bar{\varepsilon} = -52.0910063$ and $\alpha = \frac{1}{3}$.

Observe that the actual estimates, given with (48), (49) and (50) correspond to the values assumed by economic agents, given by (44), (45) and (46). Consequently, the beliefs are internally consistent, ie the beliefs affect the data generating process and at the same time find support in the data. Clearly, *rational* economic agents do not have any incentive to revise[10] their beliefs.

Furthermore, again, note that the *parameters* $\hat{\rho}$, $\hat{\hat{\bar{K}}}$ and $\sigma_{l,v}^2$ play a dual role. First of all, they determine the *actual* dynamics but at the same time are determined by the *actual* dynamics. In other words, these are fully endogenous parameters, and their values are determined in equilibrium. Formally, the values of $\hat{\rho}$, $\hat{\hat{\bar{K}}}$ and $\sigma_{l,v}^2$ correspond to a fixed point. This fixed point, however, is a fixed point in the space of reality shaping beliefs of *rational* agents. Finally, observe that, given the description of the model, parameters $\hat{\rho}$, $\hat{\hat{\bar{K}}}$ and $\sigma_{l,v}^2$ reflect *presupposed* actions of *naive* agents who in fact do not exist. Therefore,

[10] Note that $\bar{x}$ is an *imaginary* parameter and is not observable. Therefore, there is no mechanical test for the consistency of $\bar{x}$. However, *rational* economic agents do check whether the value of $\bar{x}$ leads to actual aggregate dynamics consistent with the perceived one.

the *parameters* such are purely imaginary but at the same time shape reality and can be retrieved from observed data.

In addition, time series data, $\{\widehat{K}_\tau\}$, can be used to construct the time series data of the error term, $\{v_{\tau+1}\}$, in equation (23). In turn, it is possible to construct the corresponding distribution of the error terms, $F_v(\,)$,, and verify that the autocorrelations of the error term are indeed zero as originally assumed, as shown in Figure 2.

The above observations indicate that indeed it is the case that imputed reality shaping behaviour of *naive* agents who do not exist remains consistent with the observables. However, to ensure that the economy is in equilibrium, we must show that *rational* agents do not have any incentive to revise their private actions given the observables.

Recall that a given *rational* agent believes that reality is described with equation (39) whereas, in fact, the true description of reality is given by (41). To be in equilibrium, a *rational* agent, $i \in [0,1]$, must be able to reconcile the known perceived law of motion with the observables determined with equation (41), which is not known to a *rational* agent, $i \in [0,1]$. Formally, data, $\{\widehat{K}_\tau\}_{\tau \le t}$, generated with the *actual* law of motion, equation (41), must satisfy the perceived law of motion, equation (39), ie we must simultaneously have:

$$\widehat{K}_{t+1} = \frac{1}{2+B\bar{x}}\left\{\sigma_{l,\varepsilon}^2 + \bar{\varepsilon} - B\frac{1-\bar{x}}{2}\sigma_{l,v}^2 + \left(1 - \frac{1-\bar{x}}{2}B\right)A\widehat{K}_t^\alpha + \frac{1-\bar{x}}{2}B^2\left(\widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right)\right)\right\}, \tag{51}$$

and

$$\widehat{K}_{t+1} = \frac{\bar{x}}{2+B\bar{x}}\left(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}\right) + \frac{1-\bar{x}}{2+B\bar{x}}\sigma_{l,v}^2 + \frac{1}{2+B\bar{x}}A\widehat{K}_t^\alpha - \frac{1-\bar{x}}{2+B\bar{x}}B\left(\widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right)\right) + \frac{1-\bar{x}}{2}\varepsilon_t. \tag{52}$$

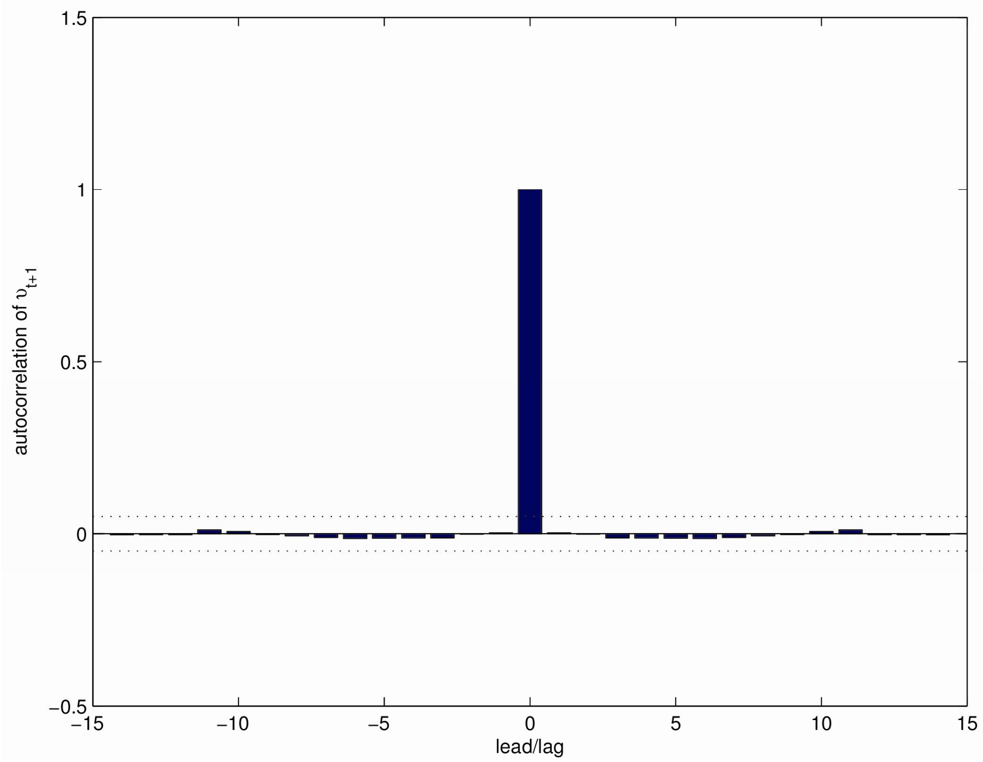

Figure 2: The Autocorrelations of v_{t+1} for T = 1000000 iterations and for $A = 50$, $B = 29.9285967$, $\bar{\varepsilon} = -52.0910063$ and $\alpha = \frac{1}{3}$.

Furthermore, given the assumption about the behaviour of *naive* agents who are just *presupposed* to exist, it must simultaneously be the case that $\hat{\rho}$ and $\hat{\hat{\bar{K}}}$ correspond to the OLS estimates on the sample data of the coefficients of the following equation:

$$\hat{K}_{t+1} = \hat{\hat{\bar{K}}} + \hat{\rho}\left(\hat{K}_t - \hat{\hat{\bar{K}}}\right) + v_{t+1}, \quad (53)$$

where v_{t+1} is uncorrelated across time.

Note that equations (51) and (53) describe the objective truth, whereas equations (52) and (53) reflect the beliefs of economic agents. Naturally, the beliefs must be in equilibrium as well. Therefore, again, for the economy to be in equilibrium, it must be the case that equations (51), (52) and (53) are simultaneously satisfied, ie the beliefs are in fact supported by observables. Moreover, error terms ε_t and v_{t+1} must have desired properties.

Can we be sure that such equilibria exist? In this paper, we answer this question affirmatively[11]. Nevertheless, we must acknowledge that our assertion that such equilibria exist relies on further technical results. Observe that the *actual* data generated with equation (51) is deterministic. Consequently, as implied by equation (52), we have:

$$\varepsilon_t = \frac{2}{1-\bar{x}}\left\{\widehat{K}_{t+1} - \frac{1}{2+B\bar{x}}\left[\bar{x}\left(\sigma_{l,\beta}^2 + \bar{\varepsilon}\right) + (1-\bar{x})\sigma_{l,v}^2 + A\widehat{K}_t^{\alpha} - (1-\bar{x})B\left(\widehat{\bar{K}} + \hat{\rho}\left(\widehat{K}_t - \widehat{\bar{K}}\right)\right)\right]\right\}, \tag{54}$$

ie error term ε_t is deterministic as well, which contradicts the beliefs of *rational* economic agents in our model that ε_t is drawn from distribution $\Psi_\varepsilon(\,)$ and is stochastic in nature. Nevertheless, it can[12] be the case that the values implied by the right-hand side of equation (54) look as if they were random even though they are truly deterministic. Moreover, as pointed out by Radunskaya (1994) and Hommes (1998), it can be the case that data dictated by the right-hand side of equation (54) can in fact be formally (statistically) indistinguishable from data generated by a purely stochastic process. Consequently, in this paper, we show that if *rational* economic agents believe that the economy is constantly being hit by stochastic and time-independent disturbances, then the actual dynamics can look as if it exactly was the case despite the fact that the actual dynamics are deterministic but sufficiently complex in nature. Formally, we constructively address the challenge posed by Grandmont (1998) who necessitated a formal basis for a coherent testing of the consistency of beliefs:

The ultimate test that this approach will have to pass, however, is that such learning equilibria *must, to be acceptable, exhibit a reasonable degree of consistency with the agents' beliefs. In this respect, one might envisage situations in which agents think that they are living in a world that is relatively simple, although subject to random (eg white noise) shocks, but in which deterministic* learning equilibria *are complex (*chaotic*) enough to make the agents; forecasting mistakes still* selffullling *in a well defined sense.*

Recall that we have already assumed, equation (47), that economic agents in our model believe that $\Psi_\varepsilon(\,)$, the distribution of ε, is such so that $\sigma_{l,\varepsilon}^2 = 396.0977657$, which allows, together with assumptions captured with equations (44), (45) and (46), us to determine the *actual* time series data. The

[11] Appendix B provides technical details that allow us to construct such equilibria.

[12] In our paper the beliefs, reflected with the value of $\bar{x}$, of *rational* agents are such so this is exactly the case in equilibrium.

actual time series data, $\{\hat{K}_\tau\}$, can be used to construct the implied with equation (54) values of the sentiment shock needed to reconcile the beliefs with reality. The implied values of ε_τ allow us to construct the empirical histogram of ε_τ, which in turn permits us to derive $\hat{\Psi}_\varepsilon(\,)$ – the data implied distribution of ε_τ, which is depicted in Figure 3.

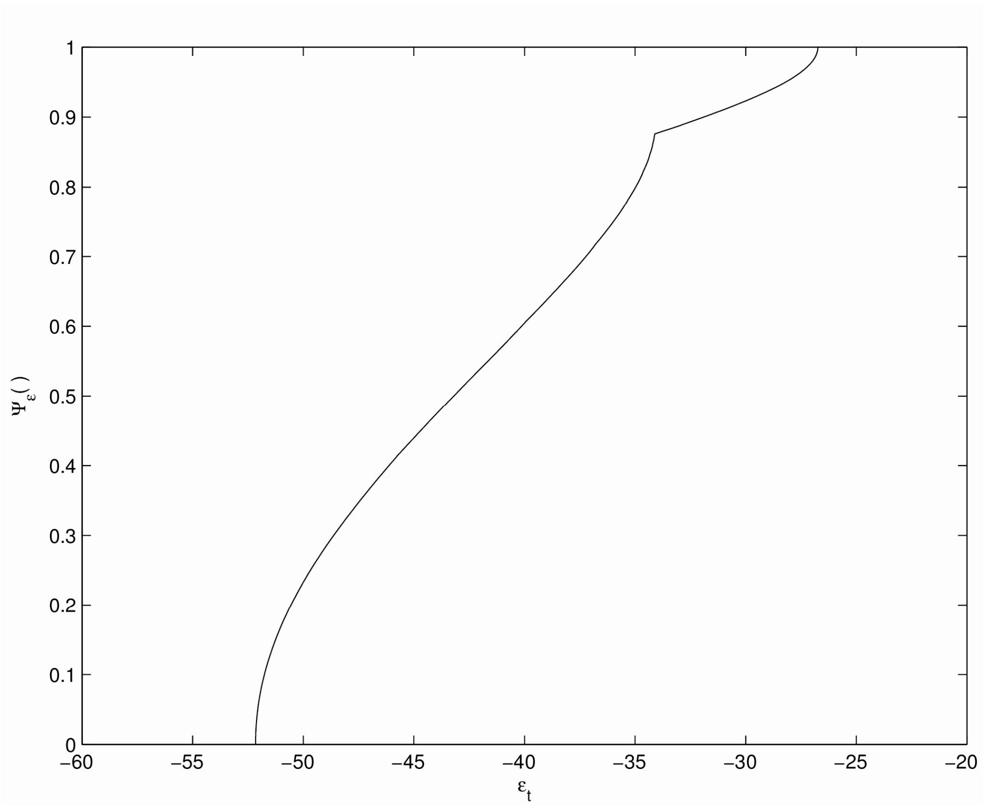

Figure 3: The Distribution Function of ε_τ for T = 1000000 iterations and for $A = 50$, $B = 29.9285967$, $\bar{\varepsilon} = -52.0910063$ and $\alpha = \frac{1}{3}$.

Now our knowledge of $\hat{\Psi}_\varepsilon(\,)$ can be used to calculate the value of $\sigma^2_{l,\varepsilon}$ implied by the observables. Specically, in this case we have:

$$\hat{\sigma}^2_{l,\varepsilon} = 396.0577987, \qquad (55)$$

which corresponds to the value originally assumed and given with equation (47). Therefore, again we can assert that beliefs of economic agents with regard to the form of $\Psi_\varepsilon(\,)$ affect the data generating process and more importantly find support in the data, ie are internally consistent. Finally, note that the autocorrelations of the values of the shocks implied with equation (54) are indeed zero (see Figure 4) as originally assumed once again confirming that the beliefs of *rational* agents (all agents) remain in equilibrium.

We have just constructively shown that agents' beliefs with regard to the degree of volatility of a given economy can affect the *actual* volatility and the *actual* volatility can be consistent with the assumed one. Specifically, as our first example shows, it can be the case that an economy remains stable if economic agents hold strong views about its stability. In particular, if economic agents believe that the economy is stable and that other agents share a similar view then the *actual* behavior, which is affected by the beliefs, of the economy conforms to the beliefs and the economy converges to a stable steady state.

On the other hand, our second example shows that a different outcome for the same values of the fundamentals is feasible as well. Specifically, if *rational* economic agents believe that the economy is volatile and is being constantly destabilised by actions of *naive* traders who are affected by sentiment shocks, then *rational* economic agents respond accordingly by building up buffer stock savings. This in turn affects the actual dynamics of the economy. We show that the change, induced by actions of *rational* agents, in the dynamics of the economy can be significant. Specifically, we constructively show that economy can become volatile in response to perceived volatility. More importantly, we argue that the actual dynamics displayed by the economy can be identical to that expected by *rational* economic agents. In other words, we show that volatility can be endogenous and constitute an outcome selected in equilibrium by self-confirming beliefs of *rational* economic agents. We provide additional numerical examples in the subsequent section.

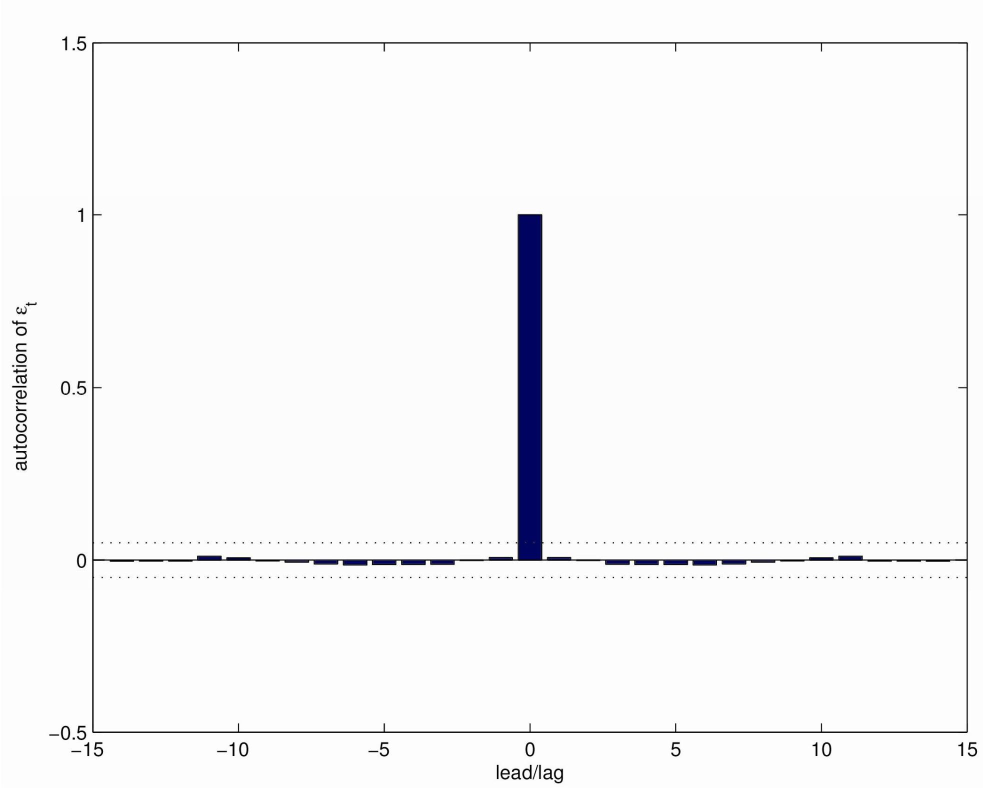

Figure 4: The Autocorrelations of ε_τ for T = 1000000 iterations and for $A = 50$, $B = 29.9285967$, $\bar{\varepsilon} = -52.0910063$ and $\alpha = \frac{1}{3}$.

The equilibrium described in our model is fragile, despite the fact that it is in fact stable in the traditional sense. The equilibrium exists only because agents are expected to not engage in any sort of out of equilibrium experimentation or thinking. Specifically, our agents could easily identify the actual law of motion if they only decided to plot K_{t+1} in terms of K_t. Furthermore, one may expect that such a plot probably should be done when *rational* economic agents perform an OLS estimation of $\widehat{\bar{K}}$ and $\hat{\rho}$ on behalf of *naive* agents. Clearly, assuming that *rational* economic agents fail to notice such a simple relationship may appear to invalidate our assumption of complete *rationality*, but formally this is not the case. As argued by Sorger (1998), the equilibrium described in the model is an example of a self-fullling mistake originally defined by Grandmont (1998). In other words, such a simple identication of the model is possible only because the original mistake (misspecication of beliefs) was made, and once it was made, the observables remain consistent with the originally misspecied beliefs. Consequently, *rational* economic agents find their original mistake self-fulfilling and do not have any incentive to change their behaviour or to experiment with other possibly simpler theories. Furthermore, we can easily eliminate the possibility of such a simple identication of the model by adding, as suggested by Grandmont (1998) and Hommes (1998), noise to the system. In our case, it suffices to assume that individual preference shocks, $\beta_{i,t}$, are affected by a stochastic factor common to all agents. Such a change would make simple identication impossible while leaving the main findings unaffected. We choose not to pursue this path for purely expositional purposes.

5. Additional results

The type of equilibria described in the previous section exist in a variety of economies. In this section, we provide some additional examples. More importantly, we show that a given economy, characterised by a given set of values of the fundamentals, can exhibit multiple equilibria. The equilibria differ with regard to the degree of volatility and naturally the degree of volatility is selected by self-confirming and equilibrium consistent beliefs of fully *rational* agents (all agents).

Let us assume at this stage that the values of the fundamentals are given by $A = 2.5$, $B = 8.741073$, $\bar{\varepsilon} = -0.0304130$ and $\alpha = \frac{1}{3}$. Furthermore, let us start our description of feasible equilibria from the simplest case. Specifically, let us assume that *rationality* is common knowledge and economic agents expect the economy to be stable. As argued earlier, in this case the accumulation, both *perceived* and *actual,* equation is given by:

$$K_{t+1} = \frac{1}{2+B}\{\bar{\varepsilon} + AK_t^{\alpha}\}. \qquad (56)$$

Naturally, in this case the economy exhibits no volatility, which verifies the initial expectations of stability and ensures that the beliefs are in equilibrium as well. The *actual* dynamics in this case is presented in Figure 5, upper left panel. Note that in this case agents do not make any errors, so the corresponding autocorrelations are not defined and not reported.

Now, imagine that the fundamentals assume the same values, ie $A = 2.5$, $B = 8.741073$, $\bar{\varepsilon} = -0.0304130$ and $\alpha = \frac{1}{3}$. Moreover, imagine that *rationality* prevails at all times, but it is not common knowledge that it does. In this case, as argued in the main part of the paper, the economy can exhibit endogenous fluctuations consistent with the private beliefs of economic agents.

Specifically, let us assume that *rational* economic agents rather than being convinced that all agents are fully *rational* believe that $\bar{x} = 0.0108779$, ie that the fraction of *rational* agents is equal to about 1.1%. Moreover, let us assume that economic agents believe that the *actual* data driven values of the equilibrium variables are given by:

$$\widehat{\widehat{K}} = 0.1655897 \tag{57}$$

and

$$\hat{\rho} = 0.2929922. \tag{58}$$

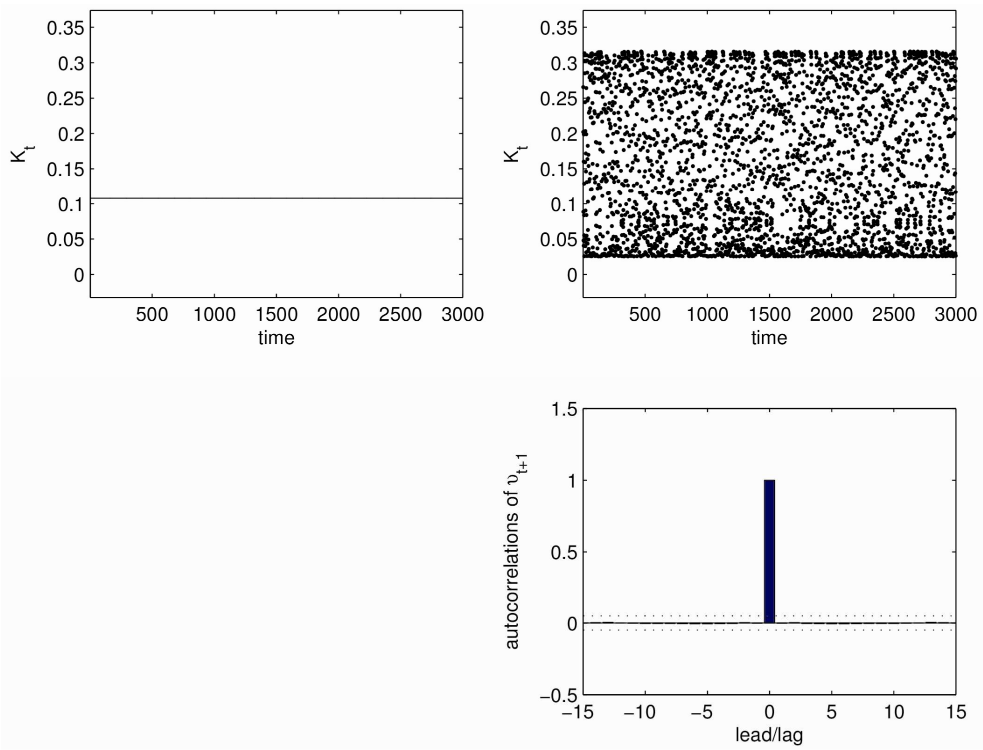

Figure 5: The *Actual* Dynamics for $T = 1000000$ iterations and the Corresponding Autocorrelations when the Perceptions of the Riskiness of the Economy Change for a Given Set of Values of the Fundamentals: $A = 2.5$, $B = 8.741073$, $\bar{\varepsilon} = -0.0304130$ and $\alpha = \frac{1}{3}$.

Moreover, assume that economic agents believe that ε_t and v_{t+1} are uncorrelated across time and that the distributions of ε_t and v_{t+1} are such so that:

$$\sigma^2_{l,v} = 0.3199794, \tag{59}$$

and[13]

$$\sigma^2_{l,\varepsilon} = -0.1870930. \tag{60}$$

In this case, there is a discrepancy between the *perceived* and the *actual* laws of motion, equations (39) and (41), respectively, but both lead to the same observational dynamics ensuring that the beliefs are always in equilibrium. Specifically, the values of the observables generated with the actual law of motion, given the expectations embodied in (57), (58), (59) and (60) can be used to estimate the *actual*, data driven, values of $\hat{\hat{K}}$, $\hat{\rho}$, $\sigma^2_{l,v}$ and $\sigma^2_{l,\varepsilon}$, which are given by:

$$\begin{gathered} \hat{\hat{\hat{K}}} = 0.1652142, \hat{\hat{\rho}} = 0.2906769, \\ \hat{\sigma}^2_{l,v} = 0.3195587 \text{ and } \hat{\sigma}^2_{l,\varepsilon} = -0.1851032. \end{gathered} \tag{61}$$

Naturally, the estimates listed in (61) confirm that the beliefs are in equilibrium. The *actual* dynamics, in this case, is presented in Figure 5, upper right panel. The corresponding autocorrelations of v_{t+1} are presented in the lower left panel.

The two examples described above reveal that multiple equilibria, with different degrees of volatility, are possible. Specifically, given the fundamentals, $A = 2.5$, $B = 8.741073$, $\bar{\varepsilon} = -0.0304130$ and $\alpha = \frac{1}{3}$, it is possible that the economy remains stable and experiences no fluctuations. Alternatively, it is possible, given the same fundamentals, that the economy fluctuates and exhibits permanent oscillations. The nature of the equilibrium is chosen by equilibrium consistent and self-confirming expectations of the agents. Consequently, the degree of volatility of the economy constitutes and outcome selected by conscious actions based on equilibrium consistent beliefs of economic agents.

[13] Recall that is not quite the variance of "; but a logarithmic transformation of a value of the moment generating function of the distribution of ": Hence, negative values of are permissible.

Now, let us consider a different example. Imagine that the fundamentals assume the following values: $A = 50$, $B = 30$, $\bar{\varepsilon} = -50.066692$ and $\alpha = \frac{1}{3}$. Furthermore, let us now consider yet another set of beliefs of economic agents. Specifically, assume that *rational* economic agents believe that $\bar{x} = 0.6337861$, ie that the fraction of *rational* agents in the population is about 63% and that time series estimates of the relevant variables are given by:

$$\hat{\hat{K}} = 1.845676, \hat{\rho} = 0.4234778,$$
$$\sigma_{l,v}^2 = 44.14143 \text{ and } \sigma_{l,\varepsilon}^2 = 272.8047. \quad (62)$$

The values listed in (62) are sufficient to determine the *actual* law of motion and in turn to generate the observables. Again, in this case as well, there is a discrepancy between the *perceived* and the *actual* laws of motion, equations (39) and (41), respectively, but both lead to the same observational dynamics ensuring that the beliefs are always in equilibrium. Specifically, the values of the observables generated with the *actual* law of motion, given the expectations embodied in (62) can be used to estimate the *actual*, data driven, values of $\hat{\hat{K}}$, $\hat{\rho}$, $\sigma_{l,v}^2$ and $\sigma_{l,\varepsilon}^2$, which are given by:

$$\hat{\hat{\hat{K}}} = 1.844378, \hat{\hat{\rho}} = 0.4234778,$$
$$\hat{\sigma}_{l,v}^2 = 44.04518 \text{ and } \hat{\sigma}_{l,\varepsilon}^2 = 272.9976. \quad (63)$$

Again, in this case as well, the estimates listed in (63) confirm that the beliefs are in equilibrium. The actual dynamics, in this case, is presented in Figure 6, upper left panel, and the corresponding autocorrelations of v_{t+1} are depicted in the lower left panel.

Now, assume that the values of the underlying fundamentals are unchanged, ie we continue to have $A = 50$, $B = 30$, $\bar{\varepsilon} = -50.066692$ and $\alpha = \frac{1}{3}$. However, agents choose to hold different beliefs than before. In particular, economic agents believe that the economy is now *riskier*. This manifests itself in a lower value of $\bar{x}$ than before. Specifically, agents believe that $\bar{x} = 0.4142131$, ie that the fraction of *rational* agents in the population in now smaller and equal to about 41%. Finally, economic agents believe that data driven values of the equilibrium variables are given by:

$$\hat{\hat{K}} = 3.084889, \hat{\rho} = 0.2803963,$$
$$\sigma_{l,v}^2 = 70.81305 \text{ and } \sigma_{l,\varepsilon}^2 = 437.3945. \quad (64)$$

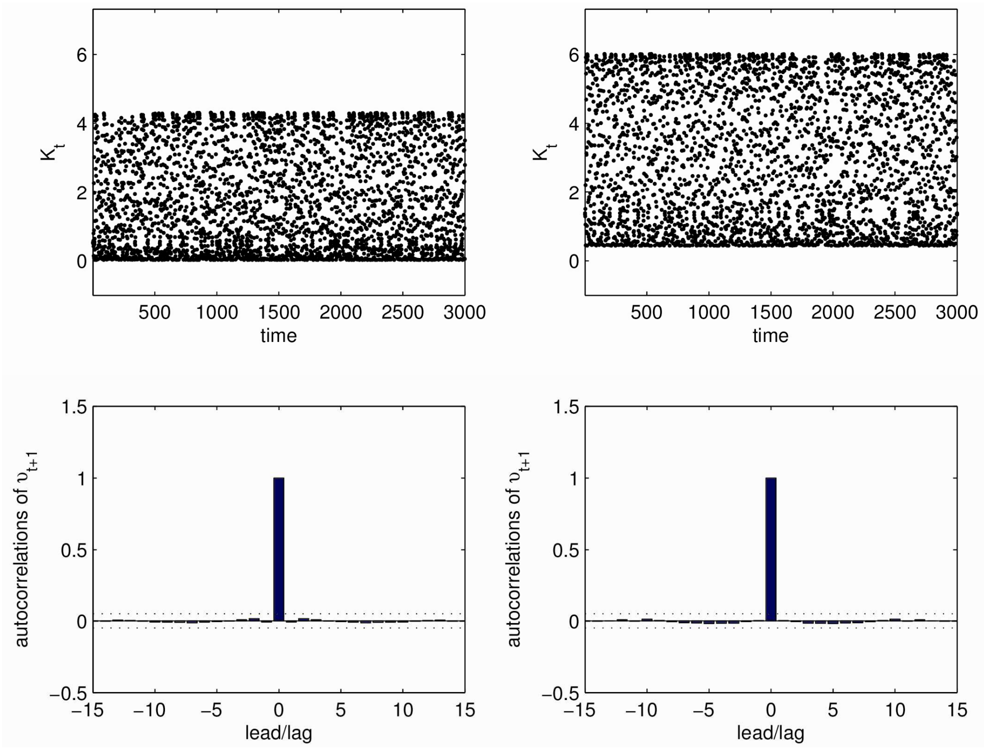

Figure 6: The *Actual* Dynamics for T = 1000000 iterations and the Corresponding Autocorrelations when the Perceptions of the Riskiness of the Economy Change for a Given Set of Values of the Fundamentals: $A = 50$, $B = 30$, $\bar{\varepsilon} = -50.066692$ and $\alpha = \frac{1}{3}$.

Naturally, the values of the fundamentals together with the beliefs of economic agent embodied in equation (64) allow us to determine the *actual* law of motion and in turn generate the observables. Again, the *actual* law of motion differs from the *perceived* law of motion. However, both lead to statistically equivalent dynamics ensuring that the beliefs of economic agents are in equilibrium. Specifically, the values of $\hat{\hat{K}}$, $\hat{\rho}$, $\sigma_{l,v}^2$ and $\sigma_{l,\varepsilon}^2$, implied with the data generated by the *actual* law of motion, are given by:

$$\begin{gathered} \hat{\hat{\hat{K}}} = 3.084944, \hat{\hat{\rho}} = 0.2800912, \\ \hat{\sigma}_{l,v}^2 = 71.40100 \text{ and } \hat{\sigma}_{l,\varepsilon}^2 = 441.3849 \end{gathered} \tag{65}$$

and naturally conform to the expected values again ensuring that the economy is equilibrium. The *actual* dynamics in this case, are presented in Figure 6, upper right hand panel, and the corresponding autocorrelations of v_{t+1} are presented in the lower right hand panel.

The above examples show that the *actual* dynamics can be shaped by self-confirming and equilibrium consistent beliefs. Furthermore, economic agents can in equilibrium select the degree of volatility of an economy. Specifically, expectations with regard to the degree of volatility of an economy influence private decisions and affect the level of bfer stock saving. This in turn affects the equilibrium process of physical capital accumulation and the resultant macro-level dynamics, which, as the above examples illustrate, can be consistent with the original expectation. Consequently, the degree of volatility of an economy can be formed by equilibrium consistent beliefs of economic agents. If agents expect the economy to be relatively stable then the resultant dynamics can be relatively stable, on the other hand, if the economic agents expect the economy to exhibit a higher level of volatility then the economy responds, without any changes in the values of the fundamentals, accordingly and the *actual* dynamics becomes more volatile verifying the beliefs and ensuring that the economy is in equilibrium.

6. Conclusions

The volatility of economic systems has been subject to a concern both from scientific and policy perspectives. Intuitively, it appears that modern economies fluctuate more than we wished and more than we can credibly account for. Not surprisingly numerous contributions have attempted to resolve the issue and to explain why modern economies are characterised by time-varying volatility. Most contributions dealing with the issue essentially ignore the most critical problem and simply assume that exogenous probability distributions govern the evolution of volatility over time. Alternatively, the volatility is endogenised, but at a cost of departures from *rationality* or under the assumption that agents, despite being statistically correct, fail to derive the link between their private actions and equilibrium dynamics. In this paper, we provide an alternative explanation that does not suffer from the standard shortcomings.

Specifically, we present a model in this paper, which allows us to understand why the degree of volatility of an economy can evolve over time even though the economy does not experience any structural changes. We argue that the degree of the volatility of an economy, rather than being imposed, can be chosen endogenously by *rational* and fully optimising agents. In particular, we illustrate in a general equilibrium framework that the perceptions of aggregate volatility formed by *rational* agents can be self-confirming, ie can result in *actual* outcomes that correspond to those expected. Specifically, we show that an economy can remain stable if economic agents expect it to be stable. On the other hand, we show that an economy can display a certain degree of volatility if economic agents expect the economy to be volatile. Naturally, at all times we ensure that the beliefs held by economic agents, with respect to the degree of volatility, remain in equilibrium themselves.

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Appendix A

In this appendix we derive the equilibrium equations describing the behaviour and reflecting the thinking of a *rational* agent, $i \in [0,1]$. First, let us start by assuming that a *rational* agent, $i \in [0,1]$, believes that the total amount invested by all *rational* agents is given by:

$$K_{t+1}^{R} = \bar{x}(n + mK_t + MK_t^{\alpha}), \tag{66}$$

where *n*, *m* and *M* are constants and $\bar{x}$ denotes the share of *rational* agents in the population.

Consequently, according to a *rational* agent, $i \in [0,1]$, the total amount invested at time *t*, $K_t = K_{t+1}^{R} + K_{t+1}^{N}$, is given by:

$$K_{t+1} = \bar{x}(n + mK_t + MK_t^{\alpha}) + \frac{1-\bar{x}}{2}\left\{\varepsilon_t + \sigma_{l,v}^2 + AK_t^{\alpha} - B\left(\hat{\hat{K}} + \hat{\rho}\left(K_t - \hat{\hat{K}}\right)\right)\right\}, \tag{67}$$

which simplifies to:

$$K_{t+1} = \bar{x}n + \frac{1-\bar{x}}{2}\left\{\sigma_{l,v}^2 - B(1-\hat{\rho})\hat{\hat{K}}\right\} + \left(\bar{x}m - \frac{1-\bar{x}}{2}B\hat{\rho}\right)K_t + \left(\bar{x}M + \frac{1-\bar{x}}{2}A\right)K_t^{\alpha} + \frac{1-\bar{x}}{2}\varepsilon_t. \tag{68}$$

Observe that all terms with exception of ε_t in the right-hand side of equation (68) are non-random. Furthermore, recall that the optimal amount saved by a *rational* agent, $i \in [0,1]$, is given by equation (17). Consequently, noting that only ε_t is, given the beliefs of *rational* agents, random with a known distribution $\Psi_{\varepsilon}(\,)$, and using some basic properties of the exponential function, we can, using equation (68), rewrite equation (17) as:

$$K_{t+1}^{i} = \frac{1}{2}\left\{\varepsilon_{i,t} + AK_t^{\alpha} - B\left\{\bar{x}(n + mK_t + MK_t^{\alpha}) + \frac{1-\bar{x}}{2}\left[\sigma_{l,v}^2 + AK_t^{\alpha} - B\left(\hat{\hat{K}} + \hat{\rho}\left(K_t - \hat{\hat{K}}\right)\right)\right]\right\} + \sigma_{l,\varepsilon}^2\right\}, \tag{69}$$

where $\sigma^2_{l,\varepsilon}$ is given by:

$$\sigma^2_{l,\varepsilon} = log\left(\int_{-\infty}^{\infty} e^{-B\frac{1-\bar{x}}{2}\varepsilon_t} d\Psi_\varepsilon(\varepsilon_t)\right). \tag{70}$$

Recall that by assumption, a *rational* agent, $i \in [0,1]$, believes that there are only $\bar{x}$ *rational* agents who share her view of the world. Consequently, according to a rational agent, $i \in [0,1]$, the amount invested by all *rational* agents at time *t* is given by $K^R_{t+1} = \bar{x}\int_0^{\bar{x}} K^j_{t*1} dj$, which, given equation (69), translates to:

$$K^R_{t+1} = \frac{1}{2}\int_0^{\bar{x}} \varepsilon_{j,t} dj + \frac{\bar{x}}{2}\Big\{AK^\alpha_t - B\Big\{\bar{x}(n + mK_t + MK^\alpha_t) + \frac{1-\bar{x}}{2}\Big[\sigma^2_{l,v} + AK^\alpha_t - B\left(\hat{\bar{K}} + \hat{\rho}\left(K_t - \hat{\bar{K}}\right)\right)\Big]\Big\} + \sigma^2_{l,\varepsilon}\Big\}. \tag{71}$$

Furthermore, recall that we have assumed that *naive* agents are aected by sentiment shocks. However, we assume that *rational* agents are not influenced by such innovations. Consequently, we have $\int_0^{\bar{x}} \varepsilon_{j,t} dj = \bar{\varepsilon}$ since $\{\varepsilon_{j,t}\}$ are truly *i.i.d*. Therefore, we can write equation (71) as:

$$K^R_{t+1} = \frac{\bar{x}}{2}\Big\{AK^\alpha_t - B\Big\{\bar{x}(n + mK_t + MK^\alpha_t) + \frac{1-\bar{x}}{2}\Big[\sigma^2_{l,v} + AK^\alpha_t - B\left(\hat{\bar{K}} + \hat{\rho}\left(K_t - \hat{\bar{K}}\right)\right)\Big]\Big\} + \sigma^2_{l,\varepsilon} + \bar{\varepsilon}\Big\}. \tag{72}$$

Now, we can simplify equation (72) by rearranging terms to:

$$K^R_{t+1} = \frac{\bar{x}}{2}\Big\{-B\Big\{\bar{x}n + \frac{1-\bar{x}}{2}\Big[\sigma^2_{l,v} - B(1-\hat{\rho})\hat{\bar{K}}\Big]\Big\} + \sigma^2_{l,\varepsilon} + \bar{\varepsilon} - B\left(\bar{x}m - \frac{1-\bar{x}}{2}B\hat{\rho}\right)K_t - B\left(\bar{x}M + \frac{1-\bar{x}}{2}A\right)K^\alpha_t\Big\}. \tag{73}$$

Matching the coeffcients of equations (66) and (73), we can find the values of *n*, *m* and *M*, which are given by:

$$\begin{gathered} n = \frac{\sigma^2_{l,\varepsilon} + \bar{\varepsilon} - B\frac{1-\bar{x}}{2}\sigma^2_{l,v} + \frac{1-\bar{x}}{2}B^2(1-\hat{\rho})\hat{\bar{K}}}{2 + B\bar{x}}, \\ m = \frac{1-\bar{x}}{2}\frac{B^2\hat{\rho}}{2+B\bar{x}}, \\ M = \frac{1-\frac{1-\bar{x}}{2}B}{2+B\bar{x}}A. \end{gathered} \tag{74}$$

Observe that the values of *n*, *m* and *M* are indeed constant as originally assumed and, thus, confirm to the beliefs of *rational* agents. Therefore, we can write the amount invested by *rational* agents as:

$$K_{t+1}^{R} = \frac{\bar{x}}{2+B\bar{x}}\left\{\sigma_{l,\varepsilon}^{2} + \bar{\varepsilon} - B\frac{1-\bar{x}}{2}\sigma_{l,v}^{2} + \left(1 - \frac{1-\bar{x}}{2}B\right)AK_{t}^{\alpha} + \frac{1-\bar{x}}{2}B^{2}\left(\widehat{\bar{K}} + \hat{\rho}\left(K_{t} - \widehat{\bar{K}}\right)\right)\right\}. \quad (75)$$

Appendix B

In this appendix, we derive formal mathematical relationships that are satisfied in the model. However, some of the relationships, even though formally valid, are not known to *rational* economic agents, but remain consistent in the observational (statistical) sense with the beliefs of *rational* economic agents.

Recall that the *perceived* law of motions is given by:

$$K_{t+1} = \frac{\bar{x}}{2+B\bar{x}}\left(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}\right) + \frac{1-\bar{x}}{2+B\bar{x}}\sigma_{l,v}^2 + \frac{1}{2+B\bar{x}}AK_t^{\alpha} - \frac{1-\bar{x}}{2+B\bar{x}}B\left(\widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right)\right) + \frac{1-\bar{x}}{2}\varepsilon_t. \quad (76)$$

where innovations $\{\varepsilon_{t+1}\}$ are aasumed to be uncorrelated across time. Furthermore, as assumed earlier, parameters $\widehat{\bar{K}}$ and $\hat{\rho}$ are OLS astimates of relationship:

$$K_{t+1} = \widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right) + v_{t+1}, \quad (77)$$

where v_{t+1} is a mean zero, time independent, error term.

Note that *rational* agents believe that preference shocks affecting *naive* agents $\{\varepsilon_t\}$ feed through the system and, as a result, impact the aggregate activity. Consequently, preference shocks affect the observables. As a result, they influence the estimates and error term $\{v_{t+1}\}$ obtained with relationship (77). In other words, *rational* economic agents understand that shocks $\{\varepsilon_t\}$ and $\{v_{t+1}\}$ can be dependent. This dependence, however, can only be verified *ex post* as the OLS regression can only be performed with a lag once K_{t+1} becomes known. In other words, *rational* economic agents cannot use the fitted value, $\{v_{t+1}\}$, of the error term at time *t* + 1 to assess the value of ε_{t+1}, which is relevant for their decision making at time *t*.

The *actual* law of motion is different from the *perceived* law of motion and is given by:

$$K_{t+1} = \frac{1}{2+B\bar{x}}\left\{\sigma_{l,\varepsilon}^2 + \bar{\varepsilon} - B\frac{1-\bar{x}}{2}\sigma_{l,v}^2 + \left(1 - \frac{1-\bar{x}}{2}B\right)AK_t^{\alpha} + \frac{1-\bar{x}}{2}B^2\left(\widehat{\bar{K}} + \hat{\rho}\left(K_t - \widehat{\bar{K}}\right)\right)\right\}. \quad (78)$$

Morover, the OLS estimates must remain valid in the sample of the observables generated with the *actual* law of motion, ie we must again have:

$$K_{t+1} = \hat{\hat{K}} + \hat{\rho}\left(K_t - \hat{\hat{K}}\right) + v_{t+1}, \tag{79}$$

Clearly, economic agents in our model believe that the economy is described with equations (76) and (77), but the objective truth is that the economy is actually described with equations (78) and (79). Naturally, from the formal perspective, agents in our model are incorrect as the *perceived* law of motion is different from the *actual* law of motion. Nevertheless, *rational* economic agents can be in equilibrium, ie they may have no incentive to revise their biased views, when the actual data generated with the *actual* law of motion fits into the *perceived* law of motion. In other words, *rational* economic agents are in fact in equilibrium when:

$$\hat{K}_{t+1} = \frac{\bar{x}}{2+B\bar{x}}\left(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}\right) + \frac{1-\bar{x}}{2+B\bar{x}}\sigma_{l,v}^2 + \frac{1}{2+B\bar{x}}AK_t^\alpha - \frac{1-\bar{x}}{2+B\bar{x}}B\left(\hat{\hat{K}} + \hat{\rho}\left(\hat{K}_t - \hat{\hat{K}}\right)\right) + \frac{1-\bar{x}}{2}\varepsilon_t, \tag{80}$$

$$\hat{K}_{t+1} = \hat{\hat{K}} + \hat{\rho}\left(\hat{K}_t - \hat{\hat{K}}\right) + v_{t+1}, \tag{81}$$

and

$$\hat{K}_{t+1} = \frac{1}{2+B\bar{x}}\left\{\sigma_{l,\varepsilon}^2 + \bar{\varepsilon} - B\frac{1-\bar{x}}{2}\sigma_{l,v}^2 + \left(1 - \frac{1-\bar{x}}{2}B\right)A\hat{K}_t^\alpha + \frac{1-\bar{x}}{2}B^2\left(\hat{\hat{K}} + \hat{\rho}\left(\hat{K}_t - \hat{\hat{K}}\right)\right)\right\}, \tag{82}$$

where $\left\{\hat{K}_t\right\}$ denotes the sample generated with the *actual* law of motion.

Again, economic agents are aware of equations (80) and (81), but they are not aware of equation (82). Nevertheless, for the equilibrium to exist, all three equations (80), (81) and (82) must be satisfied. In what follows, we manipulate the above equations to identify more informative relationships between the equilibrium variables. Our manipulations are formally valid, but cannot be done by economic agents within the model, as they do not know that equation (82) holds.

Equating the left-hand sides of equations (80) and (82) and then rearranging the terms yields:

$$\varepsilon_t = \frac{1}{2+B\bar{x}}\left\{2\left(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}\right) - (2+B)\sigma_{l,v}^2 - B\left[A\hat{K}_t^{\alpha} - (B+2)\left(\hat{\hat{K}} + \hat{\rho}\left(\hat{K}_t - \hat{\hat{K}}\right)\right)\right]\right\}. \quad (83)$$

Similarly, equating the left-hand sides of equations (81) and (83) and then rearranging the terms leads to:

$$v_{t+1} = \frac{1}{2(2+B\bar{x})}\left\{2\left(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}\right) - B(1-\bar{x})\sigma_{l,v}^2 - \left(2-(1-\bar{x})\right)B\left[A\hat{K}_t^{\alpha} - (B+2)\left(\hat{\hat{K}} + \hat{\rho}\left(\hat{K}_t - \hat{\hat{K}}\right)\right)\right]\right\}. \quad (84)$$

Observe that the *actual* law of motion, equation (82), is purely deterministic; therefore, observables, the values of $\{\hat{K}_\tau\}$, are nonrandom. Consequently, we must, given equations (83) and (84), conclude that ε_t and v_t are nonrandom as well, which formally invalidates our assumptions and formally prevents our equilibrium from being constructed. Nevertheless, the *actual* law of motion can exhibit chaotic dynamics, and it can be the case that the observables $\{\hat{K}_t\}$ can look as if they were random. Consequently, both ε_t and v_{t+1} can look as if they were random. Furthermore, it can be the case that both ε_t and v_{t+1} are uncorrelated across time. Thus, both, see Radunskaya (1994) and Hommes (1998), can be perceived as random. Formally, we can say that both ε_t and v_{t+1} can be indistinguishable from random processes from a purely statistical perspective. Therefore, *rational* agents in our model can be convinced that their beliefs conform to observables and, thus, remain in equilibrium. Furthermore, note that equations (83) and (84) actually reveal that ε_t and v_{t+1} are in fact dependent[14]. However, this knowledge cannot be used by economic agents in real time as estimates of v_{t+1} obtained with the observables are available one period after they are needed.

Let K^* denote the steady state level of capital stock, assuming that it exists, implied by the *actual* law of motion. Naturally, we must have:

$$K^* = \frac{1}{2+B\bar{x}}\left\{\sigma_{l,\varepsilon}^2 + \bar{\varepsilon} - B\frac{1-\bar{x}}{2}\sigma_{l,v}^2 + \left(1 - \frac{1-\bar{x}}{2}B\right)A(K^*)^{\propto} + \frac{1-\bar{x}}{2}B^2\left(\hat{\hat{K}} + \hat{\rho}\left(K^* - \hat{\hat{K}}\right)\right)\right\}, \quad (85)$$

[14] Operationally, agents may fail to notice that due to the presence of rounding errors.

Now, by subtracting equation (85) from equation (78) and dividing by K^*, we can establish that:

$$\frac{K_{t+1}}{K^*} - 1 = \frac{2-(1-\bar{x})B}{2(2+B\bar{x})} A(K^*)^{\propto-1} \left[\left(\frac{K_t}{K^*}\right)^{\propto} - 1\right] + \frac{(1-\bar{x})B^2}{2(2+B\bar{x})} \hat{\rho} \left(\frac{K_t}{K^*} - 1\right), \qquad (86)$$

Now, let $z_t = \frac{K_t}{K^*}$, and equation (86) can be simplified to:

$$z_{t+1} = 1 + \frac{2-(1-\bar{x})B}{2(2+B\bar{x})} A(K^*)^{\propto-1}[(z_t)^{\propto} - 1] + \frac{(1-\bar{x})B^2}{2(2+B\bar{x})} \hat{\rho}(z_t - 1). \qquad (87)$$

We want economic agents to consider ε_t and v_{t+1} to be random. This can only happen when the values of $\{K_t\}$ look random and this can happen when the *actual* law of motion exhibits chaotic dynamics, ie when equation (87) exhibits chaotic dynamics. This last requirement, together with the requirement that ε_t and v_{t+1} be uncorrelated across time, imposes restrictions on the coefficients of equation (87). In other words, we must *choose*[15] the coefficients of equation (87) to ensure that the implied dynamics of $\{z_t\}$ have the required properties. Note that the values of $\{z_t\}$ are observable. Furthermore, given the definition of z_t we can use $\{z_t\}$ to identify $\hat{\rho}$ and $\mu_z = \frac{\hat{\bar{K}}}{K^*}$ by running an OLS regression on:

$$z_{t+1} = \mu_z + \hat{\rho}(z_t - 1) + v_{\tau+1}. \qquad (88)$$

Recall again that $\hat{\rho}$ and μ_z are in fact endogenous. Both shape the *actual* law of motion, equation (87), but also must be consistent with observables generated by the *actual* law of motion and, in particular, must be equal to the OLS estimates on the observed sample of the coefficients in the relationship (88).

Let us rewrite equations (83) and (84) as:

$$\varepsilon_t = \frac{2\left(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}\right) - (2+B)\sigma_{l,v}^2}{2+B\bar{x}} - \frac{2BK^*}{2-(1-\bar{x})B} \left[\frac{2-(1-\bar{x})B}{2(2+B\bar{x})} A(K^*)^{\propto-1}(z_t)^{\propto} - \left(1 - \frac{(1-\bar{x})B^2}{2(2+B\bar{x})}\right)\left(\mu_z + \hat{\rho}(z_t - \mu_z)\right)\right], \qquad (89)$$

[15] In fact, in this model we are able to choose the coe-cients as they depend, in particular, on imaginary parameter x; which can assume any value between zero and one.

and

$$v_{t+1} = \frac{2(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}) - B(1-\bar{x})\sigma_{l,v}^2}{2+B\bar{x}} + K^* \left[\frac{2-(1-\bar{x})B}{2(2+B\bar{x})} A(K^*)^{\propto-1}(z_t)^{\propto} - \left(1 - \frac{(1-\bar{x})B^2}{2(2+B\bar{x})}\right)\left(\mu_z + \hat{\rho}(z_t - \mu_z)\right)\right]. \quad (90)$$

Let us denote the coefficients of equation (87) with:

$$\theta = \frac{2-(1-\bar{x})B}{2(2+B\bar{x})} A(K^*)^{\propto-1}, \quad (91)$$

and

$$\eta = \frac{(1-\bar{x})B^2}{2(2+B\bar{x})}\hat{\rho}. \quad (92)$$

Now equations (87), (89) and (90) can be written as:

$$z_{t+1} = 1 + \theta((z_t)^{\propto} - 1) + \eta(z_t - 1), \quad (93)$$

$$\varepsilon_t = \frac{2(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}) - (2+B)\sigma_{l,v}^2}{2+B\bar{x}} - \frac{2BK^*}{2-(1-\bar{x})B}\left[\theta - \left(1 - \frac{\eta}{\hat{\rho}}\right)\left(\mu_z + \hat{\rho}(z_t - \mu_z)\right)\right], \quad (94)$$

and

$$v_{t+1} = \frac{2(\sigma_{l,\varepsilon}^2 + \bar{\varepsilon}) - B(1-\bar{x})\sigma_{l,v}^2}{2+B\bar{x}} + K^* \left[\theta(z_t)^{\propto} - \left(1 - \frac{\eta}{\hat{\rho}}\right)\left(\mu_z + \hat{\rho}(z_t - \mu_z)\right)\right]. \quad (95)$$

Let us reiterate that θ and η must be such so that $\{z_t\}$ defined with equation (93) looks random despite being deterministic and $\{\varepsilon_t\}$ and $\{v_{t+1}\}$ are uncorrelated across time. Finally, it must be the case that the OLS estimates of the coefficients of equation (88) correspond to the values of $\hat{\rho}$ and μ_z used in the expressions describing ε_t and v_{t+1}. These restrictions on the values of θ and η are just preconditions for the equilibrium to exist.

Note that once θ and η are fixed, then $\{z_t\}$ is defined uniquely. Moreover, $\hat{\rho}$ and μ_z are defined uniquely. Consequently, the following variable:

$$y_t = \theta(z_t)^{\propto} - \left(1 - \frac{\eta}{\hat{\rho}}\right)\left(\mu_z + \hat{\rho}(z_t - \mu_z)\right), \quad (96)$$

is well defined and looks as if it were random. Let $f_y(|\theta, \eta)$ denote the *pdf* of $\{y_t\}$. Note that $f_y(|\theta, \eta)$ is now well-defined.

Clearly, we can, but agents in the model cannot, write that:

$$\varepsilon_t = \frac{2(\sigma_{l,\varepsilon}^2+\bar{\varepsilon})-(2+B)\sigma_{l,v}^2}{2+B\bar{x}} - \frac{2BK^*}{2-(1-\bar{x})B} y_t, \quad (97)$$

and

$$v_{t+1} = \frac{2(\sigma_{l,\varepsilon}^2+\bar{\varepsilon})-B(1-\bar{x})\sigma_{l,v}^2}{2+B\bar{x}} + K^* y_t. \quad (98)$$

Now, recalling the definitions of $\sigma_{l,\varepsilon}^2$ and $\sigma_{l,v}^2$, equations (38) and (29), respectively, and noting the relationships between $f_y(\,)$, $f_\varepsilon(\,)$ and $f_v(\,)$, and introducing changes of variables implied by equations (97) and (98), we can write:

$$\sigma_{l,\varepsilon}^2 = log\left(\int_{-\infty}^{\infty} e^{-B\frac{1-\bar{x}}{2}\left(\frac{2(\sigma_{l,\varepsilon}^2+\bar{\varepsilon})-(2+B)\sigma_{l,v}^2}{2+B\bar{x}} - \frac{2BK^*}{2-(1-\bar{x})B} y_t\right)} f_y(y_t|\theta,\eta)dy_t\right), \quad (99)$$

and

$$\sigma_{l,v}^2 = log\left(\int_{-\infty}^{\infty} e^{-B\left(\frac{2(\sigma_{l,\varepsilon}^2+\bar{\varepsilon})-B(1-\bar{x})\sigma_{l,v}^2}{2+B\bar{x}} + K^* y_t\right)} f_y(y_t|\theta,\eta)dy_t\right) \quad (100)$$

We can now use some basic properties of the exponential function to write:

$$\sigma_{l,\varepsilon}^2 = -\frac{B(1-\bar{x})[2(\sigma_{l,\varepsilon}^2+\bar{\varepsilon})-(2+B)\sigma_{l,v}^2]}{2(2+B\bar{x})} + log\left(\int_{-\infty}^{\infty} e^{\frac{(1-\bar{x})B^2K^*}{2-(1-\bar{x})B} y_t} f_y(y_t|\theta,\eta)dy_t\right), \quad (101)$$

and

$$\sigma_{l,v}^2 = -B\,\frac{2(\sigma_{l,\varepsilon}^2+\bar{\varepsilon})-B(1-\bar{x})\sigma_{l,v}^2}{2(2+B\bar{x})} + log\left(\int_{-\infty}^{\infty} e^{-BK^* y_t} f_y(y_t|\theta,\eta)dy_t\right) \quad (102)$$

Equations (101) and (102) form a system that allows us to solve for $\sigma_{l,\varepsilon}^2$ and $\sigma_{l,v}^2$ as:

$$\sigma_{l,\varepsilon}^2 = \frac{1}{2}\left\{(1-\bar{x})B\left[log\left(\int_{-\infty}^{\infty} e^{-BK^* y_t} f_y(y_t|\theta,\eta)dy_t\right) - \bar{\varepsilon}\right] + [2-(1-\bar{x})B]log\left(\int_{-\infty}^{\infty} e^{\frac{(1-\bar{x})B^2K^*}{2-(1-\bar{x})B}y_t} f_y(y_t|\theta,\eta)dy_t\right)\right\}, \tag{103}$$

and

$$\sigma_{l,v}^2 = log\left(\int_{-\infty}^{\infty} e^{-BK^* y_t} f_y(y_t|\theta,\eta)dy_t\right) - \frac{B}{B+2}\left[\bar{\varepsilon} + log\left(\int_{-\infty}^{\infty} e^{\frac{(1-\bar{x})B^2K^*}{2-(1-\bar{x})B}y_t} f_y(y_t|\theta,\eta)dy_t\right)\right]. \tag{104}$$

Furthermore, note that equation (85) defining K^* can be written as:

$$K^* = \frac{2(\sigma_{l,\varepsilon}^2+\bar{\varepsilon}) - B(1-\bar{x})\sigma_{l,v}^2}{2(2+B\bar{x})} + K^*\left[\frac{2-(1-\bar{x})B}{2(2+B\bar{x})}A(K^*)^{\propto-1} + \frac{(1-\bar{x})B^2}{2(2+B\bar{x})}\left(\mu_z + \hat{\rho}(1-\mu_z)\right)\right], \tag{105}$$

which, using equations (91) and (92), can be written as:

$$K^* = \frac{1}{1-\theta-\frac{\eta}{\hat{\rho}}\left(\mu_z+\hat{\rho}(1-\mu_z)\right)}\frac{2(\sigma_{l,\varepsilon}^2+\bar{\varepsilon}) - B(1-\bar{x})\sigma_{l,v}^2}{2(2+B\bar{x})}. \tag{106}$$

We can now use equations (103) and (104) to simplify equations (106) further. Specifically, we have:

$$K^* = \frac{1}{1-\theta-\frac{\eta}{\hat{\rho}}\left(\mu_z+\hat{\rho}(1-\mu_z)\right)}\frac{1}{2+B}\left[\bar{\varepsilon} + log\left(\int_{-\infty}^{\infty} e^{\frac{(1-\bar{x})B^2K^*}{2-(1-\bar{x})B}y_t} f_y(y_t|\theta,\eta)dy_t\right)\right]. \tag{107}$$

Furthermore, noting that $\frac{\eta}{\theta\hat{\rho}} = \frac{(1-\bar{x})B^2}{2(2+B\bar{x})}A(K^*)^{\propto-1}$ we can rewrite the above equation as:

$$K^* = \frac{1}{1-\theta-\frac{\eta}{\hat{\rho}}\left(\mu_z+\hat{\rho}(1-\mu_z)\right)}\frac{1}{2+B}\left[\bar{\varepsilon} + log\left(\int_{-\infty}^{\infty} e^{\frac{\eta}{\theta\hat{\rho}}A(K^*)^{\propto}y_t} f_y(y_t|\theta,\eta)dy_t\right)\right], \tag{108}$$

where of course θ and η are given by:

$$\theta = \frac{2-(1-\bar{x})B}{2(2+B\bar{x})} A(K^*)^{\propto-1}, \quad (109)$$

and

$$\eta = \frac{(1-\bar{x})B^2}{2(2+B\bar{x})} \hat{\rho}. \quad (110)$$

Now, equation (109) implies that:

$$K^* = \frac{1}{\theta} \frac{2-(1-\bar{x})B}{2(2+B\bar{x})} A(K^*)^{\propto}, \quad (111)$$

and equation (110) leads to:

$$1 - \frac{\eta}{\hat{\rho}} = \frac{[2-(1-\bar{x})](2+B)}{2(2+B\bar{x})}. \quad (112)$$

Combining equations (108), (111) and (112) allows us to establish that:

$$\frac{1}{\theta}\left(1-\frac{\eta}{\hat{\rho}}\right) A(K^*)^{\propto} = \frac{1}{1-\theta-\frac{\eta}{\hat{\rho}}(\mu_z+\hat{\rho}(1-\mu_z))}\left[\bar{\varepsilon} + log\left(\int_{-\infty}^{\infty} e^{\frac{\eta}{\theta\hat{\rho}}A(K^*)^{\propto} y_t} f_y(y_t|\theta,\eta) dy_t\right)\right], \quad (113)$$

which implicitly defines[16] $A(K^*)^{\propto}$.

Equations (108), (109) and (110) define the equilibrium for a given set of coefficients, θ and η. Note that *A*, *B* and $\propto$ reflect the values of the fundamentals of the economy and are given as such. On the other hand $\bar{x}$ is an *imaginary* parameter and its value can be adjusted as long as it remains within [0, 1] interval. Finally, K^* is free as long as it remains positive. Clearly, from a technical point of view, the system that defines the equilibrium, equations (108), (109) and (110), is a system, for given values of θ and η, of three equations with two unknowns ($\bar{x}$, K^*) and as such typically does not have a solution. Therefore, equilibrium does not exist for a given set of parameters θ and η. Naturally, it may exist if we allow parameters θ and η to vary as well.

Even if it is the case that the equilibrium does not exist for given θ and η and given *A*, *B* and $\propto$, it can still be the case that the equilibrium exists for special values of the underlying parameters. In other words, we can ask a much more modest question. Is there an economy for which the equilibrium described

[16] Assuming a solution exists.

with equations (108), (109) and (110) exists? Obviously, once we decide to free A, B and $\propto$, then the system of equations (108), (109) and (110) becomes a system of three equations with five unknowns and as such typically has a solution. We provide examples of such solutions in the main part of the text. In fact there will be a multitude of economies with equilibria that are of interest to us in this paper. In other words, for a given θ and η we are always able to provide examples of economies with equilibria that exhibit the desired properties. However, a randomly chosen economy will not have an equilibrium that is of interest to us for a given set of values of θ and η. Nevertheless, an equilibrium with the desired properties may still exist if we allow for an adjustment[17] in θ and η. Finally, even if our results are not generic for a given set of values of θ and η it may still be the case that our results hold on a non-degenerate set of values of the underlying parameters. We substantiate our claim in Appendix C.

[17] An adjustment must be done carefully as the values of θ and η must induce desired properties of $\{z_t\}$ defined with equation (93).

Appendix C

In this Appendix we describe the set of values of the underlying parameters that allow us to construct the equilibria of interest in this paper. Recall that the underlying properties of the equilibrium dynamics hinge on the properties of the following recursive equation:

$$z_{t+1} = 1 + \theta((z_t)^{\propto} - 1) + \eta(z_t - 1). \tag{114}$$

Specifically, it is a precondition for our results to obtain that the values[18] of θ and η be such so that the values of z_t dictated by equation (114) display chaotic dynamics.

Observe that our choice of θ and η determines the actual dynamics of $\{z_t\}$. Moreover, once θ and η are fixed and the dynamics of $\{z_t\}$ become known, then the coefficients of the following specification:

$$z_{t+1} = \mu_t + \hat{\rho}(z_t - \mu_t) + v_{t+1} \tag{115}$$

can be retrieved, given the assumptions, with a simple OLS technique. Consequently, our choice of θ and η determines $\hat{\rho}$ and μ_t.

Furthermore, we want economic agents in our model to be convinced that the observables confirm their beliefs. This, in particular, requires that the error terms v_{t+1} and ε_t be uncorrelated across time. This, in turn, is ensured by requiring that the following, see Appendix B, auxiliary variable:

$$y_t = \theta(z_t)^{\propto} - \left(1 - \frac{\eta}{\hat{\rho}}\right)\left(\mu_z + \hat{\rho}(z_t - \mu_z)\right) \tag{116}$$

be uncorrelated across time.

Clearly, to ensure feasibility of our results, we must find such values of θ and η and the resultant values of $\hat{\rho}$ and μ_t, so that $\{z_t\}$ are chaotic and $\{y_t\}$ are uncorrelated across time. It turns out that such parameters exist as demonstrated in Figure 7.

Our choice of θ and η allows us to determine $f_y(|\theta, \eta)$, the implied *pdf* of y_t, and, in turn, given the values of A and $\bar{\varepsilon}$, the steady state value of the capital stock, which is implicitly given by:

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[18] We always keep the value of at $\frac{1}{3}$.

$$\frac{1}{\theta}\left(1-\frac{\eta}{\hat{\rho}}\right)A(K^*)^{\propto} = \frac{1}{1-\theta-\frac{\eta}{\hat{\rho}}\left(\mu_z+\hat{\rho}(1-\mu_z)\right)}\left[\bar{\varepsilon}+log\left(\int_{-\infty}^{\infty} e^{\frac{\eta}{\theta\hat{\rho}}A(K^*)^{\propto}y_t} f_y(y_t|\theta,\eta)dy_t\right)\right]. \quad (117)$$

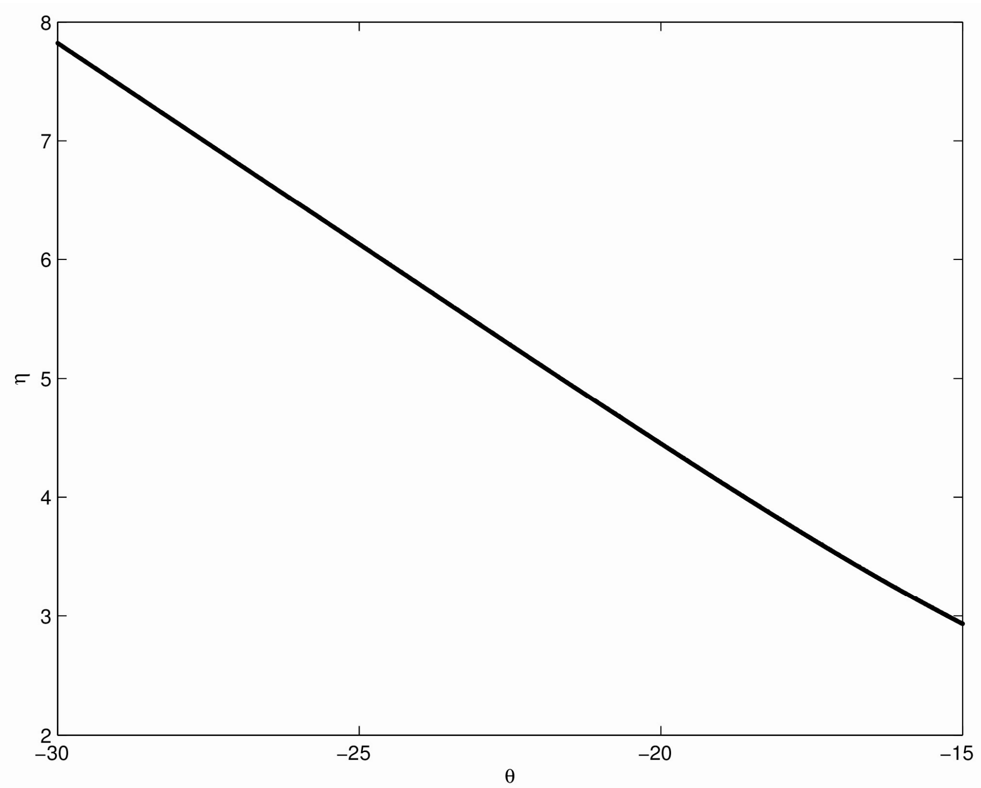

Figure 7. An Example of the Values of Parameters θ and η that lead to the Desired Dynamics of $\{z_t\}$.

Now, having found K^*, we can use the definitions of θ and η, equations (109) and (110), respectively, to establish that:

$$B = \frac{A(K^*)^{\propto-1}}{\theta}\left(1-\frac{\eta}{\hat{\rho}}\right)-2, \quad (118)$$

and

$$\bar{x} = \frac{B^2-\frac{4\eta}{\rho}}{B^2+B\frac{4\eta}{\rho}}. \quad (119)$$

Recall that equation (117) determines $A(K^{*})^{\propto}$, ie it determines K^{*} up to a scaling factor. Consequently, equations (118) and (119) determine the values of B and $\bar{x}$ needed for the equilibrium to exist given A, and K^{*}, and given specific choices of θ and η.

Naturally, the above reasoning does not yet prove that results presented in this paper are robust. Formally, the value of B is given and cannot be adjusted to ensure that an equilibrium with the desired properties exists. Therefore, so far we have shown that there exist economies with equilibria described in this paper. We can do more, however. Recall that $\bar{\varepsilon}$, B, A and $\propto$ constitute the fundamentals in our model. In other words, the values of $\bar{\varepsilon}$, B, A and $\propto$ are given and cannot be altered. However, $\bar{x}$ is different in nature as it reflects the beliefs of *rational* economic agents and as such is purely *imaginary*. Consequently, it can assume any value between 0 and 1. Therefore, we can ask a different question. Is it possible, given $\bar{\varepsilon}$, B, A and $\propto$, to find a value of $\bar{x} \in [0,1]$, such that the equilibrium has the desired properties. Unfortunately, for a random choice of $\bar{\varepsilon}$, B, A and $\propto$ the answer remains negative, ie the equilibria described in this paper are not generic. However, if we restrict the values of B to a subset of a real line, then the answer to the question is positive. In fact, for a given choice[19] of B there are many equilibria as Figure 8 indicates. Consequently, we have just argued that the equilibria described in this paper, if not generic in nature, exist for a non-degenerate set of the values of the parameters, ie for a wide variety of the values of the parameters. Figure 9 shows combinations of $\bar{x}$, $\bar{\varepsilon}$ and B that lead to equilibria of interest when A = 50.

[19] We implicitly hold the values of the remaining fundamentals $\bar{\varepsilon}$, A and α constant. A similar exercise can be done for a fixed value of B and allowing for example A to vary.

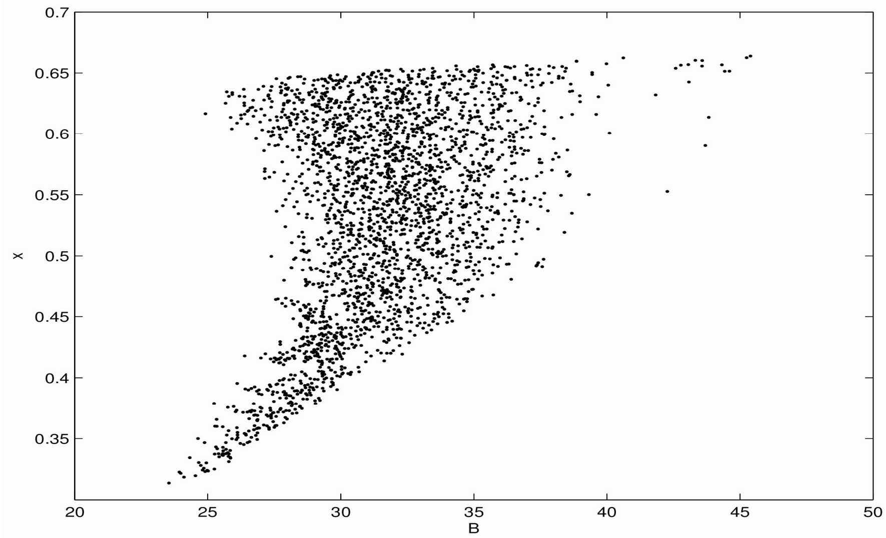

Figure 8. The Values of $\bar{x}$ that ensure that the Dynamics of $\{z_t\}$ have the Des ired Properties given the Value of Fundamental B for $A = 50$, $\bar{\varepsilon} = -52.0910063$ and $\alpha = \frac{1}{3}$.

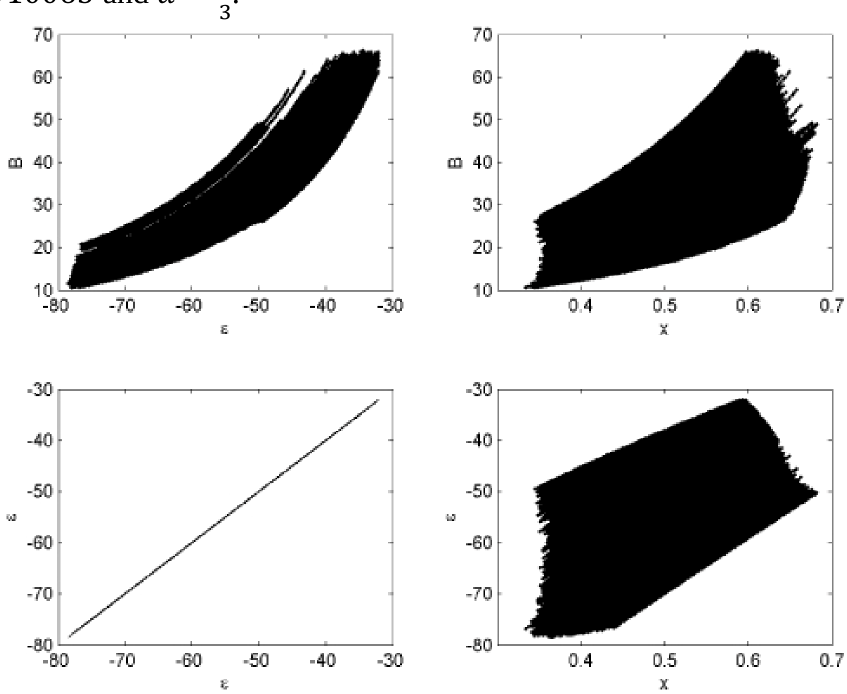

Figure 9. Combinations of $\bar{x}$, $\bar{\varepsilon}$ and B that lead to equilibria of interest when A = 50.