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On the possibility of artificial uncertainty in the leader-follower game

Abstract

In this paper we study equilibrium dynamics in a simple duopoly game. We show that the beliefs about the structure of the game influence the actual dynamics. Furthermore, we argue that beliefs induced dynamics can be complex and can in equilibrium be consistent with the underlying beliefs. We illustrate that the beliefs can be consistent even if the underlying beliefs do not correspond to the objective truth.

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1. Introduction

In this paper we examine the role of perceptions about an economy and their impact on the actual dynamics of the economy. In modern economic modeling we allow for a time dimension, and at the same time we expect the decision makers to be sophisticated in their optimal intertemporal choices. Naturally, this implies that agents' beliefs must be modeled accordingly. In particular, the beliefs must be a part of equilibrium and must be model consistent.

Traditionally, the consistency between the model and the beliefs has been achieved by demanding economic agents be rational, i.e., that their private beliefs correspond to the objective truth. In this paper we provide an example of a situation where the consistency is achieved even when the beliefs at the individual level do not correspond to the objective truth. Nevertheless, at the same time the observed dynamics is consistent with the underlying beliefs.

Our result fundamentally follows from a very basic observation that multiple models can give rise to a given observed dynamics. Furthermore, contrary to the most positions in the literature we rely on an obvious observation that the underlying model is endogenous and its shape is in fact determined by the beliefs held by economic agents. We constructively show that it is possible that a belief-shaped model generates dynamics consistent with the beliefs even though, objectively speaking, the model itself differs from the one deemed correct by economic agents.

In the paper we present a very simple case based on a duopoly game. In the game one of the players behaves always in the same manner. Specifically, the player assumes that she is a follower and always acts accordingly. At the same time we assume that the other player is not sure about the nature of the game played. In fact the player believes that there are two possibilities: the game could be of the Stackelberg form, or it could be of the Cournot form. In other words, the player does not know whether the other player perceives her as a leader or an equal. In her assessment at some periods the outcome corresponds to the outcome in a Stackelberg game, and in some to the outcome in a Cournot game. The leader assigns subjectively probabilities to the two perceived possibilities, and then updates her beliefs about the nature of the game and the state of the demand. Updated beliefs change the behavior and, in turn, impact the actual dynamics of the game. We show that the observed dynamics can correspond to the beliefs, and confirm that the equilibrium indeed exists and is in fact belief-driven.

The basic idea utilized in the paper stems from the prior contributions of Sorger (1998), Hommes (1998), and Dudek (2012), who show that the observed dynamics not only is endogenous, but shaped by the beliefs that need not correspond to the objective truth. In this paper, we show that similar type of equilibria can arise in a very basic and otherwise familiar setup. Furthermore, we show that the resulting dynamics can be very rich and, in fact, it can be indistinguishable from purely stochastic dynamics.

The paper is organized in five sections. In the following section we outline the basic setup of the game. Then we introduced perceived uncertainty into the system. In Section 4 we discuss the possibility of the consistency of beliefs and analyze the ensuing dynamics. Section 5 concludes.

2. Basic setup

In this section we outline the traditional leader-follower game of quantity competition. There are two producers who compete in a given market by setting quantities produced. For simplicity, and without any loss of generality, let us assume that the marginal cost is constant and equal to 0, i.e., in our setup effectively producers are revenue maximizing firms. Furthermore, let us assume that the demand is given by

$$P = 24 - Q. \tag{1}$$

In the traditional setup we have two firms, one being in a privileged position, the leader, and the other, the follower, that reacts to the choices of the leader. It is straightforward to establish that the best response of the follower to the choice of the leader is simply given by

$$Q_F = 12 - \frac{1}{2}Q_L. \tag{2}$$

Now, given the best response of the follower, one can quickly find the best response of the leader. Specifically, the profit of the leader is given by

$$\pi_L = PQ_L = (24 - Q)Q_L, \tag{3}$$

which of course, given that $Q = Q_F + Q_L$, translates to

$$\pi_L = PQ_L = (24 - Q_L - Q_F)Q_L.$$

A sophisticated leader, taking into account the reaction function of the follower, can establish that her profit is

$$\pi_L = PQ_L = \left(24 - Q_L - \left(12 - \tfrac{1}{2}Q_L\right)\right)Q_L,$$

which naturally reduces to

$$\pi_L = \left(12 - \tfrac{1}{2}Q_L\right)Q_L,$$

and implies that profit maximizing quantity is given by $Q_L = 12$, and in turn, given the reaction of the follower, equation (2), that $Q_F = 6$. Naturally, in this case the total supply is equal to $Q = Q_F + Q_L = 12 + 6 = 18$, and implies that the equilibrium price is equal to $P = 24 - Q = 24 - 18 = 6$.

The equilibrium is static. The price is always equal to 6 and the quantity produced is equal to 18. A simple repetition of the game does not lead to any interesting dynamics. The equilibrium point (18, 6) will continue to appear indefinitely along the equilibrium path.

3. Perceived uncertainty

In the traditional setup the roles of the two producers are given and known to all parties. In this section we introduce a modification. Specifically, we assume that there is a possibility that one of the players is uncertain about his status. In fact, we assume that the follower simply takes her role as given and always behaves in the standard manner, simply reacting to the choice of the leader. On the other hand, we assume that the leader is uncertain about his role. Specifically, we assume that the leader believes that there is only a chance q that the follower treats her as a leader, and the chance $1 - q$ that the follower does not consider the leader as such, and simply chooses her quantity simultaneously without any particular considerations to the choice of the leader. We want to know that the assumed probabilities q and $1 - q$ reflect only the perceptions of the leader and do not conform to the objective truth as in reality the follower is always a follower, i.e., in reality we have $q = 1$.

Furthermore, let us assume that the leader, but not the follower, believes that the demand itself is stochastic. Specifically, the leader believes that the demand is of the form

$$P = 24 + \varepsilon - Q, \tag{4}$$

where ε denotes a shock with mean μ.

Note that given our assumptions we can always state that the reaction function of the follower is always the same and simply given by equation (2). Moreover, the profit of the leader is given with equation (3). However, given our assumptions the leader does not know whether the follower actually is going to act as a follower, i.e., the leader does not know whether the game played has the Cournot or Stackelberg form. Therefore, the leader does not know whether she should take the quantity produced by the other producer as given, or should use the correct reaction function, equation (2), of the follower. Given the perceived uncertainty of the leader her rational assessment of the quantity produced by the follower is given by

$$EQ_F = qQ_F^S + (1-q)Q_F^C, \tag{5}$$

where Q_F^C denotes the quantity that corresponds to the Cournot game, and Q_F^S reflects the quantity in the Stackelberg game. Naturally, the leader will take Q_F^C as given in her decision making, but at the same time will correctly assess the value of Q_F^S in line with the relevant reaction function. Note that in this case in line with the perceptions of the leader the reaction function is not quite given with condition (2) as the leader believes that the demand is stochastic. In fact, according to the leader the reaction function now takes the form

$$Q_F = 12 + \frac{1}{2}\mu - \frac{1}{2}Q_L. \tag{6}$$

Now, we can express the expected profit of the leader as

$$E\pi_L = E[(24 + \varepsilon - Q_L - Q_F)Q_L], \tag{7}$$

which simplifies, given (5), to

$$E\pi_L = (24 + \mu - Q_L - Q_F - qQ_F^S - (1-q)Q_F^C)Q_L,$$

Which, noting (6), reduces to

$$E\pi_L = \left(\left(1 - \frac{q}{2}\right)(24 + \mu - Q_L) - (1-q)Q_F^C\right)Q_L.$$

Now, we can differentiate the above condition to get the optimal quantity produced by the leader, which is given by

$$Q_L = \frac{1}{2}(24+\mu) - \frac{1-q}{2\left(1-\frac{q}{2}\right)} Q_F^C.$$

Note that the above reaction function of the leader has been derived under the assumption that the leader is not sure about the true status of the follower and simply takes Q_F^C as given, i.e. we have $\frac{\partial Q_F^C}{\partial Q_L} = 0$. Nevertheless, the objective truth is different. The follower always acts as a follower, and her true choice is always the same, i.e., we have

$$Q_F^C = Q_F^S = 12 - \frac{1}{2} Q_L.$$

Observe that in the above condition there is no μ, as we assume that the follower always objectively assesses the demand, which is always given with (1). By combining the above conditions we can establish that the actual quantity produced by each of the producers is given by

$$Q_L = \frac{24}{3-q} + \mu \frac{2-q}{3-q} \tag{8}$$

and

$$Q_F = \frac{2-q}{3-q}\left(12 - \frac{\mu}{2}\right), \tag{9}$$

which implies that the total output is given by

$$Q = \frac{4-q}{3-q} 12 + \frac{2-q}{3-q}\frac{\mu}{2}. \tag{10}$$

The equilibrium price in this case is given by (note that objectively there is no shock)

$$P = 24 - Q = \frac{2-q}{3-q} 12 - \frac{2-q}{3-q}\frac{\mu}{2}. \tag{11}$$

Conditions (10) and (11) reflect the true values of the equilibrium price and quantity, and as such will be observed along the equilibrium path. However, according to the perceptions of the leader the equilibrium values are given by different equations. Specifically, the leader believes that in periods when the follower acts as a follower then the actual quantity is given by

$$Q^S = Q_L + Q_F^S = \frac{3}{4}(24+\mu), \tag{12}$$

i.e., it is equal to the quantity that would be supplied in a Stackelberg game when the demand is given with condition (4). Naturally, in this case, given the beliefs of the leader the equilibrium price is given by

$$P^S = 24 + \varepsilon - Q^S = 6 + \varepsilon - \frac{3}{4}\mu. \qquad (13)$$

Furthermore, recall that the leader believes that with probability $1 - q$ the follower does not recognize her as the leader. Consequently, the leader believes that with chance $1 - q$ essentially a Cournot game is played and in that case the overall quantity is given by

$$Q^C = Q_1^C + Q_2^C = \frac{2}{3}(24 + \mu). \qquad (14)$$

In this case the market price, again according to the leader, is simply given with

$$P^C = 24 + \varepsilon - Q^C = 8 + \varepsilon - \frac{2}{3}\mu. \qquad (15)$$

In summary, in reality there is always a single outcome given with conditions (10) and 11). However, the leader believes otherwise. In her mind there are two possibilities. The outcome could be given with conditions (12) and (13), which happens with probability q, and with conditions (14) and (15), which occurs with probability $1 - q$. Could it be the case that despite holding incorrect beliefs the leader finds the actual equilibrium dynamics to be supportive of her beliefs. In other words, could it be the case that the perceptions of the leader are in fact consistent with the actual equilibrium dynamics. We examine the issue next.

4. Consistency

Note that from the formal perspective the leader is permanently wrong as her description of reality involves two distinct states of nature (the Cournot outcome and the Stackelberg outcome) whereas the actual dynamics is given with two simple conditions (10) and 11). Can we expect that nevertheless the beliefs of the leader can be sustained in equilibrium, i.e., can we expect that the leader never realizes that her perceptions of reality are incorrect?

Let us now assume that producer can observe only the price level, P, and not the aggregate quantity produced, Q. Such a case would naturally arise when marginal costs are stochastic and only individually known[1]. Therefore, the leader can only learn about the reality by observing P. Could the path of P generated with condition (11) correspond to the path consistent with the beliefs of the leader – conditions (13) and (15)?

Note that the leader makes in fact two mistakes. First, she is not sure about the nature of the game, and secondly, she believes that the demand is stochastic and affected by disturbance ε even though the actual demand is always fixed and given with condition (1). In other words, the leader believes that the demand is affected by a disturbance, which itself could follow a complicated process. Accordingly, let $f_t(\varepsilon)$ denote the prior pdf of ε on time t. Naturally, now we have $\mu_t = \int \varepsilon f_t(\varepsilon) d\varepsilon$ and the equilibrium true price is given by

$$P_t = \frac{2-q}{3-q} 12 - \frac{2-q}{3-q}\frac{\mu_t}{2}. \tag{16}$$

Note that, as expected, the beliefs of the leader feed into the actual dynamics as P_t is a function of μ_t.

Imagine that at time t the leader actually observes price P_t. In his judgment a given value of P_t is consistent with two scenarios. First, it could be an outcome in a Cournot game when the demand disturbance is given by

$$\varepsilon_t^C = P_t - 8 + \frac{2}{3}\mu_t, \tag{17}$$

which in her judgement happens with probability $1 - q$.

Alternatively, a given value P_t could be consistent with an outcome of a Stackelberg game where the demand disturbance is given by

$$\varepsilon_t^S = P_t - 6 + \frac{3}{4}\mu_t, \tag{18}$$

which according to the beliefs of the leader occurs with probability q.

Naturally, having observed the actual price and knowing about the two scenarios, the leader can update her beliefs about the distribution of ε_t. In particular, the leader can assess the value of the posterior mean of ε_t, i.e., the prior mean for period $t + 1$ in line with

[1] Recall that we have already assumed for pure analytic convenience that marginal costs are zero. Naturally, we could relax this assumption by allowing for stochastic marginal costs and, thus, unobservable Q. We choose not to do it to simplify algebra.

$$\mu_{t+1} = \varepsilon_t^C(1-q) + \varepsilon_t^S q = P_t - 8 + 2q + \frac{8+q}{12}\mu_t. \qquad (19)$$

Observe now that we have a recursive system. The price level P_t depends on μ_t, condition (16), and at the same time μ_{t+1} depends both on P_t and μ_t, condition (19). The evolution of such a recursive system can be very rich and in fact depends on q. Recall, that q reflects beliefs of the leader with regard to the nature of the game being played. In that sense q is not real but purely imaginary. In fact, from a purely modeling perspective q is a free parameter, which can assume any value. Moreover, we can even assume that q is stochastic and revealed to the leader every period. Specifically, assuming that $g(q)$ is the pdf of q, and assuming that in each period q is drawn from the corresponding distribution and revealed to the leader, we can easily notice that now from the perspective of the leader equations (16) and (19) that describe the actual dynamics are random as well, which further enriches the dynamics. The dynamics now can be very rich and can in fact be stochastic. Furthermore, now equations (17) and (18) determine the implied values of the perceived disturbance ε, which, combined with the perceived uncertainty of the nature of the game and now randomness of q, allow us to construct the distribution of ε and the corresponding pdf, $f(\varepsilon)$. However, the underlying pdf was also perceived and not real as the shocks to demand in fact do not exist. Thus, we can identify the reconstructed pdf based on the observables with the one originally assumed ensuring that we attain a consistent equilibrium.

5. Conclusions

In this paper we illustrate that dynamics observed along the equilibrium path can be belief-based. Furthermore, we show that the observed equilibrium path can be consistent with the underlying beliefs even if the underlying beliefs are objectively unfounded. The findings of this paper stem from two observations. First, we note that beliefs feed into the system and shape the actual dynamics. Secondly, we note that a given system can generate actual dynamics consistent with that generated by a totally different and independent system.

Our findings suggest that in reality economic agents need not be able to fully eliminate uncertainty as their beliefs and ensuing actions can be a source of the observed uncertainty. Uncertainty can be thus endogenous and can come from the system, and need not reflect the physical reality.

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